Chapter 2

Finite Element Approximation

2.1 Piecewise linear approximation

One-dimensional piecewise linear approximation Let us approximate one-dimensional function $f(x)$ over region $\mathrm{P}_i\mathrm{P}_j$ in one-dimensional space. The function takes values f_i, f_j at points P_i, P_j. Let x_i, x_j be coordinates of points P_i, P_j. Let P be any point of which coordinate is given by x. Let us introduce the following two functions

$$N_{i,j}(x) = \frac{\mathrm{PP}_j}{\mathrm{P}_i\mathrm{P}_j} = \frac{x_j - x}{x_j - x_i} \tag{2.1.1a}$$

$$N_{j,i}(x) = \frac{\mathrm{P}_i\mathrm{P}}{\mathrm{P}_i\mathrm{P}_j} = \frac{x - x_i}{x_j - x_i} \tag{2.1.1b}$$

Noting that

$$N_{i,j}(x) = \begin{cases} 1 & x = x_i \\ 0 & x = x_j \end{cases}, \qquad N_{j,i}(x) = \begin{cases} 0 & x = x_i \\ 1 & x = x_j \end{cases}$$

linear approximation of function $f(x)$ over region $\mathrm{P}_i\mathrm{P}_j$ is described as follows:

$$L_{i,j}(x) = f_i\, N_{i,j}(x) + f_j\, N_{j,i}(x). \tag{2.1.2}$$

This function is linear since both $N_{i,j}(x)$ and $N_{j,i}(x)$ are linear. Also, this function satisfies

$$L_{i,j}(x_i) = f_i\, N_{i,j}(x_i) + f_j\, N_{j,i}(x_i) = f_i$$
$$L_{i,j}(x_j) = f_i\, N_{i,j}(x_j) + f_j\, N_{j,i}(x_j) = f_j$$

concluding that function $L_{i,j}(x)$ provides one-dimensional approximation over $\mathrm{P}_i\mathrm{P}_j$.

Two-dimensional piecewise linear approximation Let us approximate two-dimensional functoin $f(x, y)$ over triangle region $\triangle\mathrm{P}_i\mathrm{P}_j\mathrm{P}_k$ in two-dimensional space. The function takes values f_i, f_j, f_k at points P_i, P_j, P_k. Let (x_i, y_i) be coordinates of point P_i, (x_j, y_j) be coordinates of point P_j, and (x_k, y_k) be coordinates of point P_k. Let $\mathrm{P}(x, y)$ be any point of which coordinate is given by (x, y). Let us introduce the following three functions

$$N_{i,j,k}(x,y) = \frac{\triangle \mathrm{PP}_j\mathrm{P}_k}{\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k} = \frac{(y_j - y_k)x - (x_j - x_k)y + (x_j y_k - x_k y_j)}{2\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k} \tag{2.1.3a}$$

$$N_{j,k,i}(x,y) = \frac{\triangle \mathrm{P}_i\mathrm{PP}_k}{\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k} = \frac{(y_k - y_i)x - (x_k - x_i)y + (x_k y_i - x_i y_k)}{2\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k} \tag{2.1.3b}$$

$$N_{k,i,j}(x,y) = \frac{\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}}{\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k} = \frac{(y_i - y_j)x - (x_i - x_j)y + (x_i y_j - x_j y_i)}{2\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k} \tag{2.1.3c}$$

where

$$2\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k = (x_i y_j - x_j y_i) + (x_j y_k - x_k y_j) + (x_k y_i - x_i y_k).$$

Noting that

$$N_{i,j,k}(x,y) = \begin{cases} 1 & \text{at } \mathrm{P}_i \\ 0 & \text{at } \mathrm{P}_j,\ \mathrm{P}_k \end{cases}$$

$$N_{j,k,i}(x,y) = \begin{cases} 1 & \text{at } \mathrm{P}_j \\ 0 & \text{at } \mathrm{P}_k,\ \mathrm{P}_i \end{cases}$$

$$N_{k,i,j}(x,y) = \begin{cases} 1 & \text{at } \mathrm{P}_k \\ 0 & \text{at } \mathrm{P}_i,\ \mathrm{P}_j \end{cases}$$

linear approximation of $f(x, y)$ over region $\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k$ is described as follows:

$$L_{i,j,k}(x,y) = f_i\, N_{i,j,k}(x,y) + f_j\, N_{j,k,i}(x,y) + f_k\, N_{k,i,j}(x,y) \tag{2.1.4}$$

This function is linear since $N_{i,j,k}(x, y)$, $N_{j,k,i}(x, y)$, and $N_{k,i,j}(x, y)$ are linear. Also, this function satisfies

$$L_{i,j,k}(x_i, y_i) = f_i, \qquad L_{i,j,k}(x_j, y_j) = f_j, \qquad L_{i,j,k}(x_k, y_k) = f_k$$

concluding that function $L_{i,j,k}(x, y)$ provides two-dimensional approximation over triangle region $\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k$.

Three-dimensional piecewise linear approximation Let us approximate three-dimensional function $f(x, y, z)$ over tetrahedron region $\Diamond \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k\mathrm{P}_l$ in three-dimensional space. The function takes values f_i, f_j, f_k, f_l at points P_i, P_j, P_k, P_l. Let (x_i, y_i, z_i) be coordinates of point P_i, (x_j, y_j, z_j) be coordinates of point P_j, (x_k, y_k, z_k) be coordinates of point P_k, and (x_l, y_l, z_l) be coordinates of point P_l. Let $\mathrm{P}(x, y, z)$ be any point of which coordinate is given by (x, y, z). Let us introduce the following four functions

$$N_{i,j,k,l}(x,y,z) = \frac{\Diamond \mathrm{PP}_j\mathrm{P}_k\mathrm{P}_l}{\Diamond \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k\mathrm{P}_l} \tag{2.1.5a}$$

$$N_{j,k,l,i}(x,y,z) = \frac{\Diamond \mathrm{P}_i\mathrm{PP}_k\mathrm{P}_l}{\Diamond \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k\mathrm{P}_l} \tag{2.1.5b}$$

$$N_{k,l,i,j}(x,y,z) = \frac{\Diamond \mathrm{P}_i\mathrm{P}_i\mathrm{PP}_l}{\Diamond \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k\mathrm{P}_l} \tag{2.1.5c}$$

$$N_{l,i,j,k}(x,y,z) = \frac{\Diamond \mathrm{P}_i\mathrm{P}_i\mathrm{P}_k\mathrm{P}}{\Diamond \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k\mathrm{P}_l} \tag{2.1.5d}$$

Then, linear approximation of $f(x, y, z)$ over region $\Diamond \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k\mathrm{P}_l$ is described as follows:

$$L_{i,j,k,l}(x,y,z) = f_i\, N_{i,j,k,l}(x,y,z) + f_j\, N_{j,k,l,i}(x,y,z) + f_k\, N_{k,l,i,j}(x,y,z) + f_l\, N_{l,i,j,k}(x,y,z)$$

(2.1.6)

This function $L_{i,j,k,l}(x, y, z)$ provides three-dimensional approximation over tetrahedron region $\Diamond \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l$.

2.2 One-dimensional finite element approximation

Strain potential and kinetic energies are formulated as integral forms over one-, two-, or three-dimensional regions. It is difficult or impossible to analytically calculate such integrals. *Finite element approximation* provides methods to calculate the integrals numerically. Finite element approximation employs divide-and-conquer approach, which is outlined as follows:

Step 1 Obtain integral form with respect to unknown functions.
Step 2 Divide the integral into a finite number of integrals over small regions.
Step 3 Approximate unknown functions to calculate integrals over small regions.
Step 4 Sum up the calculated integrals over small regions.

Recall that strain potential energy of a one-dimensional soft robot is given by eq. (1.5.3), that is:

$$U = \int_0^L \frac{1}{2} E \varepsilon^2 A \, \mathrm{d}x = \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x} \right)^2 \mathrm{d}x \tag{2.2.1}$$

This integral U includes one unknown function $u(x)$, which should be obtained. The above integral over region $[\,0,\, L\,]$ can be divided into, for example, integrals over four small regions:

$$\int_0^L = \int_{x_1}^{x_2} + \int_{x_2}^{x_3} + \int_{x_3}^{x_4} + \int_{x_4}^{x_5}$$

Applying piecewise linear approximation, we analytically or numerically calculate individual integrals over small regions, resulting that we can obtain integral U.

Finite element approximation of strain potential energy Let us detail the above procedure. Divide region $[\,0,\, L\,]$ into a finite number of small regions. Here we divide the region into four equal regions. Width of the small regions is $h = L/4$. End points of small regions are referred to as *nodal points*. Here we have five nodal points. Let us describe the nodal points as $x_1 = 0$, $x_2 = h$, $x_3 = 2h$, $\cdots$, $x_5 = L$. Dividing integral interval $[\,0,\, L\,]$ into small regions, we have

$$U = \int_{x_1}^{x_2} \frac{1}{2} EA \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x + \int_{x_2}^{x_3} \frac{1}{2} EA \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x + \cdots + \int_{x_4}^{x_5} \frac{1}{2} EA \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \mathrm{d}x. \tag{2.2.2}$$

We apply piecewise linear approximation (eq. (2.1.2)) to function $u(x)$ over small region $[\,x_i,\, x_j\,]$. Piecewise linear approximation of the function is described as follows:

$$u(x) = u_i \, N_{i,j}(x) + u_j \, N_{j,i}(x), \quad x \in [\,x_i,\, x_j\,] \tag{2.2.3}$$

where u_i, u_j represent displacements at nodal points $\mathrm{P}(x_i)$, $\mathrm{P}(x_j)$. Through this approximation, function $u(x)$ can be described by five parameters u_1, u_2, $\cdots$, u_5.

Let us substitute the above piecewise linear approximation into individual integrals over small regions. For sake of simplicity, assume that Young's modulus E and cross-sectional area A are constants. Substituting piecewise linear approximation given in eq. (2.2.3) into integral over small region $[\,x_i,\,x_j\,]$, we have

$$\int_{x_i}^{x_j} \frac{1}{2} EA \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x = \frac{1}{2} \begin{bmatrix} u_i & u_j \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

(see Problem 5 in Chapter 2). Consequently, we have

$$\begin{aligned} U = &\frac{1}{2} \begin{bmatrix} u_1 & u_2 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &+ \frac{1}{2} \begin{bmatrix} u_2 & u_3 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} + \cdots \\ &+ \frac{1}{2} \begin{bmatrix} u_4 & u_5 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_4 \\ u_5 \end{bmatrix} \end{aligned}$$

which directly yields

$$U = \frac{1}{2} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}.$$

Introducing *nodal displacement vector*

$$\boldsymbol{u}_{\mathrm{N}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \tag{2.2.4}$$

and *stiffness matrix*

$$K = \frac{EA}{h} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \tag{2.2.5}$$

strain potential energy is described by the following quadratic form:

$$U = \frac{1}{2} \boldsymbol{u}_{\mathrm{N}}^{\top} K \boldsymbol{u}_{\mathrm{N}}. \tag{2.2.6}$$

Note that K is a band matrix.

Let us calculate strain potential energy of a one-dimensional soft robot with non-uniform cross-sectional area. Let function $A(x)$ denote the cross-sectional area at $\mathrm{P}(x)$. Assume that Young's modulus E is constant. Recalling that $\mathrm{d}u/\mathrm{d}x$ takes a constant value $(-u_i + u_j)/h$ in small region $[\,x_i,\,x_j\,]$, strain potential energy over the region is given as

$$\begin{aligned} &\int_{x_i}^{x_j} \frac{1}{2} EA(x) \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x = \frac{1}{2} E \left(\frac{-u_i + u_j}{h}\right)^2 \int_{x_i}^{x_j} A(x)\,\mathrm{d}x \\ &= \frac{1}{2} \begin{bmatrix} u_i & u_j \end{bmatrix} \frac{E}{h^2} \begin{bmatrix} V_{i,j} & -V_{i,j} \\ -V_{i,j} & V_{i,j} \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix} \end{aligned}$$

where

$$V_{i,j} = \int_{x_i}^{x_j} A(x)\,\mathrm{d}x \tag{2.2.7}$$

represents the volume of the three-dimensional region specified by small region $[\,x_i,\,x_j\,]$. Thus, when region $[\,0,\,L\,]$ is divided into four small regions, stiffness matrix is described as follows:

$$K = \frac{E}{h^2}\begin{bmatrix} V_{1,2} & -V_{1,2} & & & \\ -V_{1,2} & V_{1,2}+V_{2,3} & -V_{2,3} & & \\ & -V_{2,3} & V_{2,3}+V_{3,4} & -V_{3,4} & \\ & & -V_{3,4} & V_{3,4}+V_{4,5} & -V_{4,5} \\ & & & -V_{4,5} & V_{4,5} \end{bmatrix}. \tag{2.2.8}$$

This matrix K is also a band matrix.

Let us reformulate the above calculation. Assume again that Young's modulus E and cross-sectional area A are constants. Potential energy stored in region $[\,x_i,\,x_j\,]$ is given by

$$U_{i,j} = \frac{1}{2}\begin{bmatrix} x_i & x_j \end{bmatrix} K_{i,j} \begin{bmatrix} x_i \\ x_j \end{bmatrix} \tag{2.2.9}$$

where

$$K_{i,j} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{2.2.10}$$

We obtain stiffness matrix K (eq. (2.2.5)) by synthesizing matrices $K_{1,2}$, $K_{2,3}$, $K_{3,4}$, and $K_{4,5}$. Let us introduce operator $\oplus$ to describe this synthesizing:

$$K = K_{1,2} \oplus K_{2,3} \oplus K_{3,4} \oplus K_{4,5} \tag{2.2.11}$$

This equation implies that summing up all contributions of $K_{1,2}$ through $K_{4,5}$ yields stiffness matrix K. Note that

$(1,1)$-th element of $K_{i,j}$ contributes to (i,i)-th element of K,
$(1,2)$-th element of $K_{i,j}$ contributes to (i,j)-th element of K,
$(2,1)$-th element of $K_{i,j}$ contributes to (j,i)-th element of K,
$(2,2)$-th element of $K_{i,j}$ contributes to (j,j)-th element of K.

We simply describe these contributions as

$(1,2)\times(1,2)$ elements of $K_{i,j}$ contribute to $(i,j)\times(i,j)$ elements of K.

Namely, operator $\times$ denotes direct product: $(1,2)\times(1,2)$ implies $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$ while $(i,j)\times(i,j)$ implies (i,i), (i,j), (j,i), (j,j).

Finite element approximation of kinetic energy Let us calculate kinetic energy of a one-dimensional soft robot given by eq. (1.6.2), that is:

$$T = \int_0^L \frac{1}{2}\rho A \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x = \int_0^L \frac{1}{2}\rho A \dot{u}^2\,\mathrm{d}x \tag{2.2.12}$$

For sake of simplicity, assume that density ρ and cross-sectional area A are constants. Dividing integral region $[\,0,\,L\,]$ into four equal regions, we have

$$T = \int_{x_1}^{x_2} \frac{1}{2}\rho A\dot{u}^2 \mathrm{d}x + \int_{x_2}^{x_3} \frac{1}{2}\rho A\dot{u}^2 \mathrm{d}x + \cdots + \int_{x_4}^{x_5} \frac{1}{2}\rho A\dot{u}^2 \mathrm{d}x \tag{2.2.13}$$

Piecewise linear approximation of function $u(x,t)$ over small region $[\,x_i,\,x_j\,]$ is described as follows:

$$u(x,t) = u_i(t)\,N_{i,j}(x) + u_j(t)\,N_{j,i}(x), \quad x \in [\,x_i,\,x_j\,] \tag{2.2.14}$$

Note that u_i, u_j depend on time t whereas functions $N_{i,j}(x)$, $N_{j,i}(x)$ are not. Differentiating the above equation with respect time t yields

$$\dot{u}(x,t) = \dot{u}_i(t)\,N_{i,j}(x) + \dot{u}_j(t)\,N_{j,i}(x), \quad x \in [\,x_i,\,x_j\,] \tag{2.2.15}$$

Applying the above equation into integral over small region $[\,x_i,\,x_j\,]$, we have

$$\int_{x_i}^{x_j} \frac{1}{2}\rho A \dot{u}^2 \mathrm{d}x = \frac{1}{2}\begin{bmatrix} \dot{u}_i & \dot{u}_j \end{bmatrix} \rho A h \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} \dot{u}_i \\ \dot{u}_j \end{bmatrix}$$

(see Problem 4 in Chapter 2). Consequently,

$$\begin{aligned} T = {} & \frac{1}{2}\begin{bmatrix} \dot{u}_1 & \dot{u}_2 \end{bmatrix} \rho A h \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \\ & + \frac{1}{2}\begin{bmatrix} \dot{u}_2 & \dot{u}_3 \end{bmatrix} \rho A h \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} + \cdots \\ & + \frac{1}{2}\begin{bmatrix} \dot{u}_4 & \dot{u}_5 \end{bmatrix} \rho A h \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} \dot{u}_4 \\ \dot{u}_5 \end{bmatrix} \end{aligned}$$

which directly yields

$$T = \frac{1}{2} \dot{\boldsymbol{u}}_{\mathrm{N}}{}^{\top} M \dot{\boldsymbol{u}}_{\mathrm{N}} \tag{2.2.16}$$

where $\dot{\boldsymbol{u}}_{\mathrm{N}} = [\,\dot{u}_1,\,\dot{u}_2,\,\cdots,\,\dot{u}_5\,]^{\top}$ and

$$M = \rho A h \cdot \frac{1}{6} \begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{bmatrix} \tag{2.2.17}$$

Matrix M is referred to as a *inertia matrix*. Note that M is a band matrix. Sum of all elements of M coincides with the total mass, implying that the inertia matrix defines its distribution. The above calculation is simply described as

$$M = M_{1,2} \oplus M_{2,3} \oplus M_{3,4} \oplus M_{4,5} \tag{2.2.18}$$

where

$$M_{i,j} = \frac{\rho A h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{2.2.19}$$

denotes a partial inertia matrix corresponding to region $[\,x_i,\,x_j\,]$.

Let us calculate kinetic energy of a one-dimensional soft robot with non-uniform cross-sectional area. Let function $A(x)$ denote the cross-sectional area at $\mathrm{P}(x)$. Assume that density ρ is constant. Kinetic energy over small region $[\,x_i,\,x_j\,]$ is then given by

$$\int_{x_i}^{x_j} \frac{1}{2}\rho A \dot{u}^2 \mathrm{d}x = \frac{1}{2}\begin{bmatrix} \dot{u}_i & \dot{u}_j \end{bmatrix} \rho \begin{bmatrix} \bar{V}_{i,j}^{i,i} & \bar{V}_{i,j}^{i,j} \\ \bar{V}_{i,j}^{i,j} & \bar{V}_{i,j}^{j,j} \end{bmatrix} \begin{bmatrix} \dot{u}_i \\ \dot{u}_j \end{bmatrix}$$

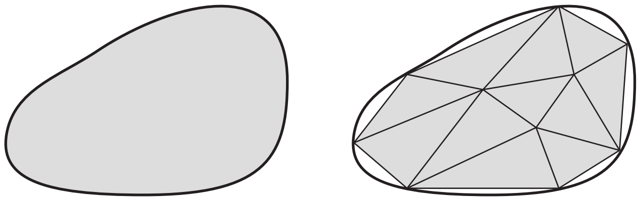

Figure 2.1: Approximation of two-dimensional region

where

$$\bar{V}_{i,j}^{i,i} = \int_{x_i}^{x_j} A(x)\{N_{i,j}(x)\}^2 \mathrm{d}x, \qquad \bar{V}_{i,j}^{j,j} = \int_{x_i}^{x_j} A(x)\{N_{j,i}(x)\}^2 \mathrm{d}x,$$
$$\bar{V}_{i,j}^{i,j} = \int_{x_i}^{x_j} A(x)\, N_{i,j}(x)\, N_{j,i}(x)\, \mathrm{d}x$$

Thus, when region $[\,0,\,L\,]$ is divided into four small regions, inertia matrix is described as follows:

$$M = \rho \begin{bmatrix} \bar{V}_{1,2}^{1,1} & \bar{V}_{1,2}^{1,2} & & & \\ \bar{V}_{1,2}^{1,2} & \bar{V}_{1,2}^{2,2} + \bar{V}_{2,3}^{2,2} & \bar{V}_{2,3}^{2,3} & & \\ & \bar{V}_{2,3}^{2,3} & \bar{V}_{2,3}^{3,3} + \bar{V}_{3,4}^{3,3} & \bar{V}_{3,4}^{3,4} & \\ & & \bar{V}_{3,4}^{3,4} & \bar{V}_{3,4}^{4,4} + \bar{V}_{4,5}^{4,4} & \bar{V}_{4,5}^{4,5} \\ & & & \bar{V}_{4,5}^{4,5} & \bar{V}_{4,5}^{5,5} \end{bmatrix}.$$

This matrix M is also a band matrix. Note that $\bar{V}_{i,j}^{i,i} + \bar{V}_{i,j}^{j,j} + 2\bar{V}_{i,j}^{i,j} = V_{i,j}$, implying that the inertia matrix defines the distribution of the total mass.

2.3 Two-dimensional finite element approximation

Strain potential energy and kinetic energy of a two-dimensional soft robot are formulated by integrals over two-dimensional region S, which is often described by an irregular shape, making analytical calculation of integrals difficult or impossible. Let us approximate two-dimensional region S (Fig. 2.1(a)) by a set of small triangles (Fig. 2.1(b)). Then, integral over two-dimensional region S can be approximated by the sum of integrals over small triangles:

$$\int_S \approx \sum_{\triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k} \int_{\triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k}$$

Here we apply piecewise linear approximation to individual integrals over small triangles so that we can analytically or numerically calculate the integrals.

Finite element approximation of kinetic energy Let us calculate kinetic energy of a two-dimensional soft robot given by eq. (1.6.4), that is:

$$T = \int_S \frac{1}{2} \rho\, \dot{\boldsymbol{u}}^{\top} \dot{\boldsymbol{u}}\, h\, \mathrm{d}S \tag{2.3.1}$$

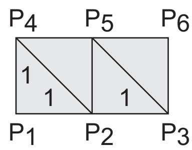

Figure 2.2: Example of rectangle region

First, we calculate integral over triangle region $\triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$:

$$T_{i,j,k} = \int_{\triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k} \frac{1}{2} \rho \, \dot{\boldsymbol{u}}^\top \dot{\boldsymbol{u}} \, h \, \mathrm{d}S \tag{2.3.2}$$

Piecewise linear approximation of function $\boldsymbol{u}$ over triangle region $\triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$ is described as follows:

$$\boldsymbol{u} = \boldsymbol{u}_i \, N_{i,j,k} + \boldsymbol{u}_j \, N_{j,k,i} + \boldsymbol{u}_k \, N_{k,i,j}. \tag{2.3.3}$$

Noting that $\boldsymbol{u}_i$, $\boldsymbol{u}_j$, $\boldsymbol{u}_k$ depend on time while $N_{i,j,k}$, $N_{j,k,i}$, $N_{k,i,j}$ do not, we have

$$\dot{\boldsymbol{u}} = \dot{\boldsymbol{u}}_i \, N_{i,j,k} + \dot{\boldsymbol{u}}_j \, N_{j,k,i} + \dot{\boldsymbol{u}}_k \, N_{k,i,j} \tag{2.3.4}$$

which directly yields

$$\dot{\boldsymbol{u}}^\top \dot{\boldsymbol{u}} =$$
$$\begin{bmatrix} \dot{\boldsymbol{u}}_i^\top & \dot{\boldsymbol{u}}_j^\top & \dot{\boldsymbol{u}}_k^\top \end{bmatrix} \begin{bmatrix} \{N_{i,j,k}\}^2 I_{2\times 2} & N_{i,j,k} N_{j,k,i} I_{2\times 2} & N_{i,j,k} N_{k,i,j} I_{2\times 2} \\ N_{i,j,k} N_{j,k,i} I_{2\times 2} & \{N_{j,k,i}\}^2 I_{2\times 2} & N_{j,k,i} N_{k,i,j} I_{2\times 2} \\ N_{i,j,k} N_{k,i,j} I_{2\times 2} & N_{j,k,i} N_{k,i,j} I_{2\times 2} & \{N_{k,i,j}\}^2 I_{2\times 2} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}_i \\ \dot{\boldsymbol{u}}_j \\ \dot{\boldsymbol{u}}_k \end{bmatrix}$$

where $I_{2\times 2}$ represents 2×2 identical matrix. For sake of simplicity, assume that density ρ and thickness h are constants. Then,

$$\begin{aligned} T_{i,j,k} &= \frac{1}{2} \begin{bmatrix} \dot{\boldsymbol{u}}_i^\top & \dot{\boldsymbol{u}}_j^\top & \dot{\boldsymbol{u}}_k^\top \end{bmatrix} \rho h \begin{bmatrix} (\triangle/6) I_{2\times 2} & (\triangle/12) I_{2\times 2} & (\triangle/12) I_{2\times 2} \\ (\triangle/12) I_{2\times 2} & (\triangle/6) I_{2\times 2} & (\triangle/12) I_{2\times 2} \\ (\triangle/12) I_{2\times 2} & (\triangle/12) I_{2\times 2} & (\triangle/6) I_{2\times 2} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}_i \\ \dot{\boldsymbol{u}}_j \\ \dot{\boldsymbol{u}}_k \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \dot{\boldsymbol{u}}_i^\top & \dot{\boldsymbol{u}}_j^\top & \dot{\boldsymbol{u}}_k^\top \end{bmatrix} \frac{\rho h \triangle}{12} \begin{bmatrix} 2I_{2\times 2} & I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}_i \\ \dot{\boldsymbol{u}}_j \\ \dot{\boldsymbol{u}}_k \end{bmatrix} \end{aligned} \tag{2.3.5}$$

where $\triangle = \triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$ (see Problem 6). Matrix

$$M_{i,j,k} = \frac{\rho h \triangle}{12} \begin{bmatrix} 2I_{2\times 2} & I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} \end{bmatrix} \tag{2.3.6}$$

is referred to as *partial inertia matrix*. Note that the sum of all blocks of matrix $M_{i,j,k}$ is equal to $\rho h \triangle \, I_{2\times 2}$, which denotes the mass of this triangular element.

Let us calculate the total kinetic energy over rectangle region $\square \mathrm{P}_1 \mathrm{P}_3 \mathrm{P}_6 \mathrm{P}_4$ shown in Fig. 2.2. This region consists of four triangle regions: $\triangle \mathrm{P}_1 \mathrm{P}_2 \mathrm{P}_4$, $\triangle \mathrm{P}_2 \mathrm{P}_3 \mathrm{P}_5$, $\triangle \mathrm{P}_5 \mathrm{P}_4 \mathrm{P}_2$, and $\triangle \mathrm{P}_6 \mathrm{P}_5 \mathrm{P}_3$. For sake of simplicity, assume that $\rho h \triangle / 12$ is constantly equal to 1. Then, partial inertia matrices are given as

$$M_{1,2,4} = M_{2,3,5} = M_{5,4,2} = M_{6,5,3} = \begin{bmatrix} 2I_{2\times 2} & I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} \end{bmatrix}.$$

Let $\boldsymbol{u}_{\mathrm{N}}$ be a collective vector consisting of all displacement vectors at nodal points:

$$\boldsymbol{u}_{\mathrm{N}} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_6 \end{bmatrix} \tag{2.3.7}$$

which is referred to as *nodal displacement vector.* The total kinetic energy is then described by a quadratic form with respect to $\dot{\boldsymbol{u}}_{\mathrm{N}}$:

$$T = \frac{1}{2}\dot{\boldsymbol{u}}_{\mathrm{N}}^{\top}\, M\, \dot{\boldsymbol{u}}_{\mathrm{N}},$$

where M is referred to as *inertia matrix.* Noting that

$$(1,2,3) \times (1,2,3) \text{ blocks of } M_{1,2,4} \text{ contribute to } (1,2,4) \times (1,2,4) \text{ blocks of } M,$$

namely,

$$\begin{array}{l} (1,1),\ (1,2),\ (1,3) \text{ blocks of } M_{1,2,4} \text{ contribute to } (1,1),\ (1,2),\ (1,4) \text{ blocks of } M, \\ (2,1),\ (2,2),\ (2,3) \text{ blocks of } M_{1,2,4} \text{ contribute to } (2,1),\ (2,2),\ (2,4) \text{ blocks of } M, \\ (3,1),\ (3,2),\ (3,3) \text{ blocks of } M_{1,2,4} \text{ contribute to } (4,1),\ (4,2),\ (4,4) \text{ blocks of } M, \end{array}$$

we find contribution of $M_{1,2,4}$ to M as follows:

$$\left[\begin{array}{c|c|c|c|c|c} 2I_{2\times 2} & I_{2\times 2} & & I_{2\times 2} & & \\ \hline I_{2\times 2} & 2I_{2\times 2} & & I_{2\times 2} & & \\ \hline & & & & & \\ \hline I_{2\times 2} & I_{2\times 2} & & 2I_{2\times 2} & & \\ \hline & & & & & \\ \hline & & & & & \end{array}\right].$$

Similarly,

$$(1,2,3) \times (1,2,3) \text{ blocks of } M_{5,4,2} \text{ contribute to } (5,4,2) \times (5,4,2) \text{ blocks of } M,$$

namely,

$$\begin{array}{l} (1,1),\ (1,2),\ (1,3) \text{ blocks of } M_{5,4,2} \text{ contribute to } (5,5),\ (5,4),\ (5,2) \text{ blocks of } M, \\ (2,1),\ (2,2),\ (2,3) \text{ blocks of } M_{5,4,2} \text{ contribute to } (4,5),\ (4,4),\ (4,2) \text{ blocks of } M, \\ (3,1),\ (3,2),\ (3,3) \text{ blocks of } M_{5,4,2} \text{ contribute to } (2,5),\ (2,4),\ (2,2) \text{ blocks of } M, \end{array}$$

we find contribution of $M_{5,4,2}$ to M as follows:

$$\left[\begin{array}{c|c|c|c|c|c} & & & & & \\ \hline & 2I_{2\times 2} & & I_{2\times 2} & I_{2\times 2} & \\ \hline & & & & & \\ \hline & I_{2\times 2} & & 2I_{2\times 2} & I_{2\times 2} & \\ \hline & I_{2\times 2} & & I_{2\times 2} & 2I_{2\times 2} & \\ \hline & & & & & \end{array}\right].$$

Summing up all contributions, we finally have

$$M = \begin{bmatrix} 2I_{2\times 2} & I_{2\times 2} & & I_{2\times 2} & & \\ I_{2\times 2} & 6I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} & 2I_{2\times 2} & \\ & I_{2\times 2} & 4I_{2\times 2} & & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & & 4I_{2\times 2} & I_{2\times 2} & \\ & 2I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} & 6I_{2\times 2} & I_{2\times 2} \\ & & I_{2\times 2} & & I_{2\times 2} & 2I_{2\times 2} \end{bmatrix}.$$

This inertia matrix M consists of 6^2 2×2 blocks and is a *sparse matrix*. We simply describe the above calculation as

$$M = M_{1,2,4} \oplus M_{2,3,5} \oplus M_{5,4,2} \oplus M_{6,5,3}. \tag{2.3.8}$$

Operator $\oplus$ works block-wise. In general, inertia matrix is described as

$$M = \bigoplus_{i,j,k} M_{i,j,k} \tag{2.3.9}$$

where i, j, k represent nodal point numbers of each triangle.

Finite element approximation of strain potential energy We apply the above calculation to strain potential energy. First, let us calculate strain potential energy stored in small triangle region $\triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$:

$$U_{i,j,k} = \int_{\triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k} \frac{1}{2} \boldsymbol{\varepsilon}^\top (\lambda I_\lambda + \mu I_\mu) \boldsymbol{\varepsilon} \, h \, \mathrm{d}S. \tag{2.3.10}$$

Piecewise linear approximation of function $\boldsymbol{u}$ over triangle region $\triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$ is described as $\boldsymbol{u} = \boldsymbol{u}_i \, N_{i,j,k} + \boldsymbol{u}_j \, N_{j,k,i} + \boldsymbol{u}_k \, N_{k,i,j}$. Introducing collective vectors $\boldsymbol{\gamma}_u = [\, u_i, \, u_j, \, u_k \,]^\top$ and $\boldsymbol{\gamma}_v = [\, v_i, \, v_j, \, v_k \,]^\top$, we find

$$\frac{\partial u}{\partial x} = \boldsymbol{a}^\top \boldsymbol{\gamma}_u, \qquad \frac{\partial u}{\partial y} = \boldsymbol{b}^\top \boldsymbol{\gamma}_u, \qquad \frac{\partial v}{\partial x} = \boldsymbol{a}^\top \boldsymbol{\gamma}_v, \qquad \frac{\partial v}{\partial y} = \boldsymbol{b}^\top \boldsymbol{\gamma}_v$$

where

$$\boldsymbol{a} = \frac{1}{2\triangle} \begin{bmatrix} y_j - y_k \\ y_k - y_i \\ y_i - y_j \end{bmatrix}, \qquad \boldsymbol{b} = \frac{-1}{2\triangle} \begin{bmatrix} x_j - x_k \\ x_k - x_i \\ x_i - x_j \end{bmatrix} \tag{2.3.11}$$

(see Problem 2). Then, strain vector is given as

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{a}^\top \boldsymbol{\gamma}_u \\ \boldsymbol{b}^\top \boldsymbol{\gamma}_v \\ \boldsymbol{b}^\top \boldsymbol{\gamma}_u + \boldsymbol{a}^\top \boldsymbol{\gamma}_v \end{bmatrix} \tag{2.3.12}$$

Substituting the above equation into eq. (2.3.10), we have

$$\begin{aligned} U_{i,j,k} &= \frac{1}{2} \begin{bmatrix} \boldsymbol{\gamma}_u^\top & \boldsymbol{\gamma}_v^\top \end{bmatrix} \lambda \begin{bmatrix} \boldsymbol{a}\boldsymbol{a}^\top & \boldsymbol{a}\boldsymbol{b}^\top \\ \boldsymbol{b}\boldsymbol{a}^\top & \boldsymbol{b}\boldsymbol{b}^\top \end{bmatrix} h\triangle \begin{bmatrix} \boldsymbol{\gamma}_u \\ \boldsymbol{\gamma}_v \end{bmatrix} \\ &+ \frac{1}{2} \begin{bmatrix} \boldsymbol{\gamma}_u^\top & \boldsymbol{\gamma}_v^\top \end{bmatrix} \mu \begin{bmatrix} 2\boldsymbol{a}\boldsymbol{a}^\top + \boldsymbol{b}\boldsymbol{b}^\top & \boldsymbol{b}\boldsymbol{a}^\top \\ \boldsymbol{a}\boldsymbol{b}^\top & 2\boldsymbol{b}\boldsymbol{b}^\top + \boldsymbol{a}\boldsymbol{a}^\top \end{bmatrix} h\triangle \begin{bmatrix} \boldsymbol{\gamma}_u \\ \boldsymbol{\gamma}_v \end{bmatrix} \end{aligned} \tag{2.3.13}$$

(see Problem 7). Then, we have

$$U_{i,j,k} = \frac{1}{2}\boldsymbol{\gamma}^\top (\lambda H_\lambda + \mu H_\mu)\, \boldsymbol{\gamma} \tag{2.3.14}$$

where

$$\boldsymbol{\gamma} = \begin{bmatrix} \boldsymbol{\gamma}_u \\ \boldsymbol{\gamma}_v \end{bmatrix}, \quad H_\lambda = \begin{bmatrix} \boldsymbol{a}\boldsymbol{a}^\top & \boldsymbol{a}\boldsymbol{b}^\top \\ \boldsymbol{b}\boldsymbol{a}^\top & \boldsymbol{b}\boldsymbol{b}^\top \end{bmatrix} h\triangle,$$
$$H_\mu = \begin{bmatrix} 2\boldsymbol{a}\boldsymbol{a}^\top + \boldsymbol{b}\boldsymbol{b}^\top & \boldsymbol{b}\boldsymbol{a}^\top \\ \boldsymbol{a}\boldsymbol{b}^\top & 2\boldsymbol{b}\boldsymbol{b}^\top + \boldsymbol{a}\boldsymbol{a}^\top \end{bmatrix} h\triangle.$$

The above equation is a quadratic form with respect to $\boldsymbol{\gamma} = [\, u_i,\, u_j,\, u_k,\, v_i,\, v_j,\, v_k \,]^\top$. Let us permutate rows and columns of H_λ and H_μ so that $U_{i,j,k}$ is described by a quadratic form with respect to $\boldsymbol{u}_{i,j,k} = [\, u_i,\, v_i,\, u_j,\, v_j,\, u_k,\, v_k \,]^\top$. Namely, let 1, 4, 2, 5, 3, 6 rows and columns of H_λ be 1, 2, 3, 4, 5, 6 rows and columns of $J_\lambda^{i,j,k}$. Similarly, let 1, 4, 2, 5, 3, 6 rows and columns of H_μ be 1, 2, 3, 4, 5, 6 rows and columns of $J_\mu^{i,j,k}$. Then, we have

$$\boldsymbol{\gamma}^\top H_\lambda \boldsymbol{\gamma} = \boldsymbol{u}_{i,j,k}^\top\, J_\lambda^{i,j,k}\, \boldsymbol{u}_{i,j,k}, \qquad \boldsymbol{\gamma}^\top H_\mu \boldsymbol{\gamma} = \boldsymbol{u}_{i,j,k}^\top\, J_\mu^{i,j,k}\, \boldsymbol{u}_{i,j,k}$$

Matrices $J_\lambda^{i,j,k}$ and $J_\mu^{i,j,k}$ are referred to as *partial connection matrices*. Once coordinates of P_i, P_j, P_k are given, we can calculate partial connection matrices $J_\lambda^{i,j,k}$ and $J_\mu^{i,j,k}$.

Finally, we find strain potential energy stored in $\triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k$:

$$U_{i,j,k} = \frac{1}{2}\boldsymbol{u}_{i,j,k}^\top\, K_{i,j,k}\, \boldsymbol{u}_{i,j,k} \tag{2.3.15}$$

where

$$K_{i,j,k} = \lambda J_\lambda^{i,j,k} + \mu J_\mu^{i,j,k} \tag{2.3.16}$$

is referred to as *partial stiffness matrix*.

Summing up all strain potential energies over small triangle regions, we obtain the total strain potential energy described as

$$U = \frac{1}{2}\boldsymbol{u}_\mathrm{N}^\top\, K\, \boldsymbol{u}_\mathrm{N} \tag{2.3.17}$$

where

$$K = \bigoplus_{i,j,k} K_{i,j,k} \tag{2.3.18}$$

is referred to as *stiffness matrix*. Assuming that Lamé's constants λ and μ are uniform over the region, stiffness matrix is described as

$$K = \bigoplus_{i,j,k} (\lambda J_\lambda^{i,j,k} + \mu J_\mu^{i,j,k}) = \lambda \bigoplus_{i,j,k} J_\lambda^{i,j,k} + \mu \bigoplus_{i,j,k} J_\mu^{i,j,k}$$

which directly yields

$$K = \lambda J_\lambda + \mu J_\mu \tag{2.3.19}$$

where

$$J_\lambda = \bigoplus_{i,j,k} J_\lambda^{i,j,k}, \qquad J_\mu = \bigoplus_{i,j,k} J_\mu^{i,j,k}$$

are referred to as *connection matrices*.

Example Let us calculate partial connection matrices of triangle $P_1P_2P_4$ shown in Fig. 2.2. Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ are given by $\boldsymbol{a} = [\,-1,\, 1,\, 0\,]^\top$ and $\boldsymbol{b} = [\,-1,\, 0,\, 1\,]^\top$. Assuming $h = 2$, we have

$$H_\lambda = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \end{array}\right], \qquad H_\mu = \left[\begin{array}{ccc|ccc} 3 & -2 & -1 & 1 & -1 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ \hline 1 & 0 & -1 & 3 & -1 & -2 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \end{array}\right]$$

Permuting rows and columns of the above matrices, we find

$$J_\lambda^{1,2,4} = \left[\begin{array}{cc|cc|cc} 1 & 1 & -1 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 & -1 \\ \hline -1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$J_\mu^{1,2,4} = \left[\begin{array}{cc|cc|cc} 3 & 1 & -2 & -1 & -1 & 0 \\ 1 & 3 & 0 & -1 & -1 & -2 \\ \hline -2 & 0 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ \hline -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{array}\right]$$

Let us calculate partial connection matrices of triangle $P_5P_4P_2$ shown in Fig. 2.2. Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ are given by $\boldsymbol{a} = [\,-1,\, 1,\, 0\,]^\top$ and $\boldsymbol{b} = [\,-1,\, 0,\, 1\,]^\top$. Thus, assuming $h = 2$, we find $J_\lambda^{5,4,2} = J_\lambda^{1,2,4}$ and $J_\mu^{5,4,2} = J_\mu^{1,2,4}$. Partial connection matrices are invariant with respect to translation displacement. As a result, under the same assumption, we have

$$J_\lambda^{1,2,4} = J_\lambda^{2,3,5} = J_\lambda^{5,4,2} = J_\lambda^{6,5,3}, \qquad J_\mu^{1,2,4} = J_\mu^{2,3,5} = J_\mu^{5,4,2} = J_\mu^{6,5,3}$$

Let us calculate connection matrices J_λ and J_μ of rectangle region $\square P_1P_3P_6P_4$ shown in Fig. 2.2. Noting that

$(1,2,3) \times (1,2,3)$ blocks of $J_\lambda^{1,2,4}$ contribute to $(1,2,4) \times (1,2,4)$ blocks of J_λ,

namely,

$(1,1)$, $(1,2)$, $(1,3)$ blocks of $J_\lambda^{1,2,4}$ contribute to $(1,1)$, $(1,2)$, $(1,4)$ blocks of J_λ,
$(2,1)$, $(2,2)$, $(2,3)$ blocks of $J_\lambda^{1,2,4}$ contribute to $(2,1)$, $(2,2)$, $(2,4)$ blocks of J_λ,
$(3,1)$, $(3,2)$, $(3,3)$ blocks of $J_\lambda^{1,2,4}$ contribute to $(4,1)$, $(4,2)$, $(4,4)$ blocks of J_λ,

we obtain contribution of $J_\lambda^{1,2,4}$ to J_λ. Noting that

$(1,2,3) \times (1,2,3)$ blocks of $J_\lambda^{5,4,2}$ contribute to $(5,4,2) \times (5,4,2)$ blocks of J_λ,

namely,

$(1,1)$, $(1,2)$, $(1,3)$ blocks of $J_\lambda^{5,4,2}$ contribute to $(5,5)$, $(5,4)$, $(5,2)$ blocks of J_λ,
$(2,1)$, $(2,2)$, $(2,3)$ blocks of $J_\lambda^{5,4,2}$ contribute to $(4,5)$, $(4,4)$, $(4,2)$ blocks of J_λ,
$(3,1)$, $(3,2)$, $(3,3)$ blocks of $J_\lambda^{5,4,2}$ contribute to $(2,5)$, $(2,4)$, $(2,2)$ blocks of J_λ,

we obtain contribution of $J_\lambda^{5,4,2}$ to J_λ. Summing up all contributions, we finally have

$$J_\lambda = \left[\begin{array}{rr|rr|rr|rr|rr|rr}
1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\
1 & 1 & -1 & 0 & & & 0 & -1 & & & & \\ \hline
-1 & -1 & 2 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & & \\
0 & 0 & 1 & 2 & -1 & 0 & 1 & 0 & -1 & -2 & & \\ \hline
 & & -1 & -1 & 1 & 0 & & & 0 & 1 & 0 & 0 \\
 & & 0 & 0 & 0 & 1 & & & 1 & 0 & -1 & -1 \\ \hline
0 & 0 & 0 & 1 & & & 1 & 0 & -1 & -1 & & \\
-1 & -1 & 1 & 0 & & & 0 & 1 & 0 & 0 & & \\ \hline
 & & 0 & -1 & 0 & 1 & -1 & 0 & 2 & 1 & -1 & -1 \\
 & & -1 & -2 & 1 & 0 & -1 & 0 & 1 & 2 & 0 & 0 \\ \hline
 & & & & 0 & -1 & & & -1 & 0 & 1 & 1 \\
 & & & & 0 & -1 & & & -1 & 0 & 1 & 1
\end{array}\right]$$

We simply describe the above calculation as

$$J_\lambda = J_\lambda^{1,2,4} \oplus J_\lambda^{2,3,5} \oplus J_\lambda^{5,4,2} \oplus J_\lambda^{6,5,3}. \tag{2.3.20}$$

Operator $\oplus$ works block-wise. Similarly, we have

$$J_\mu = J_\mu^{1,2,4} \oplus J_\mu^{2,3,5} \oplus J_\mu^{5,4,2} \oplus J_\mu^{6,5,3}, \tag{2.3.21}$$

which yields

$$J_\mu = \left[\begin{array}{rr|rr|rr|rr|rr|rr}
3 & 1 & -2 & -1 & & & -1 & 0 & & & & \\
1 & 3 & 0 & -1 & & & -1 & -2 & & & & \\ \hline
-2 & 0 & 6 & 1 & -2 & -1 & 0 & 1 & -2 & -1 & & \\
-1 & -1 & 1 & 6 & 0 & -1 & 1 & 0 & -1 & -4 & & \\ \hline
 & & -2 & 0 & 3 & 0 & & & 0 & 1 & -1 & -1 \\
 & & -1 & -1 & 0 & 3 & & & 1 & 0 & 0 & -2 \\ \hline
-1 & -1 & 0 & 1 & & & 3 & 0 & -2 & 0 & & \\
0 & -2 & 1 & 0 & & & 0 & 3 & -1 & -1 & & \\ \hline
 & & -2 & -1 & 0 & 1 & -2 & -1 & 6 & 1 & -2 & 0 \\
 & & -1 & -4 & 1 & 0 & 0 & -1 & 1 & 6 & -1 & -1 \\ \hline
 & & & & -1 & 0 & & & -2 & -1 & 3 & 1 \\
 & & & & -1 & -2 & & & 0 & -1 & 1 & 3
\end{array}\right]$$

Matrices J_λ and J_μ are sparse matrices.

2.4 Three-dimensional finite element approximation

Strain potential energy and kinetic energy of a three-dimensional soft robot are formulated by integrals over three-dimensional region V, which is often described by an irregular shape, making analytical calculation of integrals difficult or impossible. Let us approximate three-dimensional region V by a set of small tetrahedra. Then, integral over three-dimensional region V can be approximated by the sum of integrals over small tetrahedra:

$$\int_V \approx \sum_{\Diamond \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l} \int_{\Diamond \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l}$$

Here we apply piecewise linear approximation so that we can analytically or numerically calculate individual integrals over small tetrahedra.

Finite element approximation of kinetic energy Let us calculate kinetic energy over tetrahedron region $\Diamond \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l$:

$$T_{i,j,k,l} = \int_{\Diamond \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l} \frac{1}{2} \rho \, \dot{\boldsymbol{u}}^\top \dot{\boldsymbol{u}} \, \mathrm{d}V \tag{2.4.1}$$

Piecewise linear approximation of function $\boldsymbol{u}$ over tetrahedron region $\Diamond \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l$ is described as follows:

$$\boldsymbol{u} = \boldsymbol{u}_i \, N_{i,j,k,l} + \boldsymbol{u}_j \, N_{j,k,l,i} + \boldsymbol{u}_k \, N_{k,l,i,j} + \boldsymbol{u}_l \, N_{l,k,i,j}. \tag{2.4.2}$$

Differentiating the above equation with respect to time t, we have

$$\dot{\boldsymbol{u}} = \dot{\boldsymbol{u}}_i \, N_{i,j,k,l} + \dot{\boldsymbol{u}}_j \, N_{j,k,l,i} + \dot{\boldsymbol{u}}_k \, N_{k,l,i,j} + \dot{\boldsymbol{u}}_l \, N_{l,k,i,j}. \tag{2.4.3}$$

For sake of simplicity, assume that density ρ is constant. Letting $I_{3\times 3}$ represent 3×3 identical matrix, we have

$$T_{i,j,k,l} = \frac{1}{2} \begin{bmatrix} \dot{\boldsymbol{u}}_i^\top & \dot{\boldsymbol{u}}_j^\top & \dot{\boldsymbol{u}}_k^\top & \dot{\boldsymbol{u}}_l^\top \end{bmatrix} \frac{\rho \Diamond}{20} \begin{bmatrix} 2I_{3\times3} & I_{3\times3} & I_{3\times3} & I_{3\times3} \\ I_{3\times3} & 2I_{3\times3} & I_{3\times3} & I_{3\times3} \\ I_{3\times3} & I_{3\times3} & 2I_{3\times3} & I_{3\times3} \\ I_{3\times3} & I_{3\times3} & I_{3\times3} & 2I_{3\times3} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}_i \\ \dot{\boldsymbol{u}}_j \\ \dot{\boldsymbol{u}}_k \\ \dot{\boldsymbol{u}}_l \end{bmatrix} \tag{2.4.4}$$

where $\Diamond = \Diamond \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l$ (see Problem 8). Matrix

$$M_{i,j,k,l} = \frac{\rho \Diamond}{20} \begin{bmatrix} 2I_{3\times3} & I_{3\times3} & I_{3\times3} & I_{3\times3} \\ I_{3\times3} & 2I_{3\times3} & I_{3\times3} & I_{3\times3} \\ I_{3\times3} & I_{3\times3} & 2I_{3\times3} & I_{3\times3} \\ I_{3\times3} & I_{3\times3} & I_{3\times3} & 2I_{3\times3} \end{bmatrix} \tag{2.4.5}$$

is referred to as *partial inertia matrix.* Note that the sum of all blocks of matrix $M_{i,j,k,l}$ is equal to $\rho \Diamond \, I_{3\times 3}$, which denotes the mass of this tetrahedron element.

Summing up all kinetic energies over small tetrahedron regions, we obtain the total kinetic energy described as

$$T = \frac{1}{2} \dot{\boldsymbol{u}}_\mathrm{N}^\top \, M \, \dot{\boldsymbol{u}}_\mathrm{N},$$

where M is referred to as *inertia matrix.*

Finite element approximation of strain potential energy We calculate strain potential energy stored in small tetrahedron region $\Diamond \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l$. Introducing collective vectors $\boldsymbol{\gamma}_u = [\, u_i, \, u_j, \, u_k, \, u_l \,]^\top$, $\boldsymbol{\gamma}_v = [\, v_i, \, v_j, \, v_k, \, v_k \,]^\top$, and $\boldsymbol{\gamma}_w = [\, w_i, \, w_j, \, w_k, \, w_k \,]^\top$, we find

$$\begin{aligned} \frac{\partial u}{\partial x} &= \boldsymbol{a}^\top \boldsymbol{\gamma}_u, & \frac{\partial u}{\partial y} &= \boldsymbol{b}^\top \boldsymbol{\gamma}_u, & \frac{\partial u}{\partial z} &= \boldsymbol{c}^\top \boldsymbol{\gamma}_u, \\ \frac{\partial v}{\partial x} &= \boldsymbol{a}^\top \boldsymbol{\gamma}_v, & \frac{\partial v}{\partial y} &= \boldsymbol{b}^\top \boldsymbol{\gamma}_v, & \frac{\partial v}{\partial z} &= \boldsymbol{c}^\top \boldsymbol{\gamma}_v, \\ \frac{\partial w}{\partial x} &= \boldsymbol{a}^\top \boldsymbol{\gamma}_w, & \frac{\partial w}{\partial y} &= \boldsymbol{b}^\top \boldsymbol{\gamma}_w, & \frac{\partial w}{\partial z} &= \boldsymbol{c}^\top \boldsymbol{\gamma}_w \end{aligned}$$

where

$$\boldsymbol{a} = \frac{1}{6\Diamond} \begin{bmatrix} -a_{j,k,l} \\ a_{k,l,i} \\ -a_{l,i,j} \\ a_{i,j,k} \end{bmatrix}, \quad \boldsymbol{b} = \frac{1}{6\Diamond} \begin{bmatrix} -b_{j,k,l} \\ b_{k,l,i} \\ -b_{l,i,j} \\ b_{i,j,k} \end{bmatrix}, \quad \boldsymbol{c} = \frac{1}{6\Diamond} \begin{bmatrix} -c_{j,k,l} \\ c_{k,l,i} \\ -c_{l,i,j} \\ c_{i,j,k} \end{bmatrix}$$

with

$$\begin{aligned}
a_{j,k,l} &= (y_j z_k - y_k z_j) + (y_k z_l - y_l z_k) + (y_l z_j - y_j z_l) \\
b_{j,k,l} &= (z_j x_k - z_k x_j) + (z_k x_l - z_l x_k) + (z_l x_j - z_j x_l) \\
c_{j,k,l} &= (x_j y_k - x_k y_j) + (x_k y_l - x_l y_k) + (x_l y_j - x_j y_l)
\end{aligned}$$

(see Problem 3). Strain vector is given as

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{a}^\top \boldsymbol{\gamma}_u \\ \boldsymbol{b}^\top \boldsymbol{\gamma}_v \\ \boldsymbol{c}^\top \boldsymbol{\gamma}_w \\ \boldsymbol{c}^\top \boldsymbol{\gamma}_v + \boldsymbol{b}^\top \boldsymbol{\gamma}_w \\ \boldsymbol{a}^\top \boldsymbol{\gamma}_w + \boldsymbol{c}^\top \boldsymbol{\gamma}_u \\ \boldsymbol{b}^\top \boldsymbol{\gamma}_u + \boldsymbol{a}^\top \boldsymbol{\gamma}_v \end{bmatrix}.$$

Then, strain potential energy stored in $\Diamond \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l$ is given by

$$U_{i,j,k,l} = \frac{1}{2} \boldsymbol{\gamma}^\top (\lambda H_\lambda + \mu H_\mu) \boldsymbol{\gamma} \tag{2.4.6}$$

where

$$\boldsymbol{\gamma} = \begin{bmatrix} \boldsymbol{\gamma}_u \\ \boldsymbol{\gamma}_v \\ \boldsymbol{\gamma}_w \end{bmatrix}, \quad H_\lambda = \begin{bmatrix} \boldsymbol{a}\boldsymbol{a}^\top & \boldsymbol{a}\boldsymbol{b}^\top & \boldsymbol{a}\boldsymbol{c}^\top \\ \boldsymbol{b}\boldsymbol{a}^\top & \boldsymbol{b}\boldsymbol{b}^\top & \boldsymbol{b}\boldsymbol{c}^\top \\ \boldsymbol{c}\boldsymbol{a}^\top & \boldsymbol{c}\boldsymbol{b}^\top & \boldsymbol{c}\boldsymbol{c}^\top \end{bmatrix} \Diamond,$$

$$H_\mu = \begin{bmatrix} 2\boldsymbol{a}\boldsymbol{a}^\top + \boldsymbol{b}\boldsymbol{b}^\top + \boldsymbol{c}\boldsymbol{c}^\top & \boldsymbol{b}\boldsymbol{a}^\top & \boldsymbol{c}\boldsymbol{a}^\top \\ \boldsymbol{a}\boldsymbol{b}^\top & 2\boldsymbol{b}\boldsymbol{b}^\top + \boldsymbol{c}\boldsymbol{c}^\top + \boldsymbol{a}\boldsymbol{a}^\top & \boldsymbol{c}\boldsymbol{b}^\top \\ \boldsymbol{a}\boldsymbol{c}^\top & \boldsymbol{b}\boldsymbol{c}^\top & 2\boldsymbol{c}\boldsymbol{c}^\top + \boldsymbol{a}\boldsymbol{a}^\top + \boldsymbol{b}\boldsymbol{b}^\top \end{bmatrix} \Diamond. \tag{2.4.7}$$

Let us permutate rows and columns of H_λ and H_μ so that $U_{i,j,k,l}$ is described by a quadratic form of $\boldsymbol{u}_{i,j,k,l} = [\, \boldsymbol{u}_i^\top, \boldsymbol{u}_j^\top, \boldsymbol{u}_k^\top, \boldsymbol{u}_l^\top \,]^\top$. Namely, let 1, 5, 9, 2, 6, 10, 3, 7, 11, 4, 8, 12 rows and columns of H_λ be 1 through 12 rows and columns of $J_\lambda^{i,j,k,l}$. Similarly, let 1, 5, 9, 2, 6, 10, 3, 7, 11, 4, 8, 12 rows and columns of H_μ be 1 through 12 rows and columns of $J_\mu^{i,j,k,l}$. Then, we have

$$\boldsymbol{\gamma}^\top H_\lambda \boldsymbol{\gamma} = \boldsymbol{u}_{i,j,k,l}^\top \, J_\lambda^{i,j,k,l} \, \boldsymbol{u}_{i,j,k,l}, \qquad \boldsymbol{\gamma}^\top H_\mu \boldsymbol{\gamma} = \boldsymbol{u}_{i,j,k,l}^\top \, J_\mu^{i,j,k,l} \, \boldsymbol{u}_{i,j,k,l}.$$

Matrices $J_\lambda^{i,j,k,l}$ and $J_\mu^{i,j,k,l}$ are referred to as *partial connection matrices*. Once coordinates of P_i, P_j, P_k, P_l are given, we can calculate partial connection matrices $J_\lambda^{i,j,k,l}$ and $J_\mu^{i,j,k,l}$.

Finally, we find strain potential energy stored in $\Diamond \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l$:

$$U_{i,j,k,l} = \frac{1}{2} \boldsymbol{u}_{i,j,k,l}^\top \, K_{i,j,k,l} \, \boldsymbol{u}_{i,j,k,l} \tag{2.4.8}$$

where

$$K_{i,j,k,l} = \lambda J_\lambda^{i,j,k,l} + \mu J_\mu^{i,j,k,l} \tag{2.4.9}$$

which is referred to as *partial stiffness matrix*.

Summing up all strain potential energies over small tetrahedron regions, we obtain the total strain potential energy described as

$$U = \frac{1}{2} \boldsymbol{u}_\mathrm{N}^\top \, K \, \boldsymbol{u}_\mathrm{N} \tag{2.4.10}$$

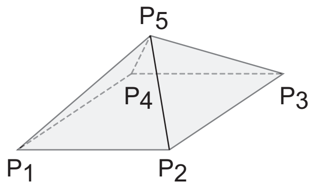

Figure 2.3: Example of regular quadrangular pyramid

where

$$K = \bigoplus_{i,j,k,l} K_{i,j,k,l} \tag{2.4.11}$$

is referred to as *stiffness matrix*. Assuming that Lamé's constants λ and μ are uniform over the region, stiffness matrix is described as

$$K = \lambda J_\lambda + \mu J_\mu \tag{2.4.12}$$

where matrices J_λ and J_μ are referred to as *connection matrices*.

Let us calculate connection matrices of a regular quadrangular pyramid (Fig. 2.3). The base of the pyramid is square $\square \mathrm{P}_1\mathrm{P}_2\mathrm{P}_3\mathrm{P}_4$ and the apex of the pyramid is P_5. Coordinates of vertices are given by $\boldsymbol{x}_1 = [0,\ 0,\ 0]^\top$, $\boldsymbol{x}_2 = [2,\ 0,\ 0]^\top$, $\boldsymbol{x}_3 = [2,\ 2,\ 0]^\top$, $\boldsymbol{x}_4 = [0,\ 2,\ 0]^\top$, and $\boldsymbol{x}_5 = [1,\ 1,\ 1]^\top$. The piramid consists of two tetraheda: $\lozenge \mathrm{P}_1\mathrm{P}_2\mathrm{P}_3\mathrm{P}_5$ and $\lozenge \mathrm{P}_3\mathrm{P}_4\mathrm{P}_1\mathrm{P}_5$.

Partial connection matrices of $\lozenge \mathrm{P}_1\mathrm{P}_2\mathrm{P}_3\mathrm{P}_5$ are as follows:

$$J_\lambda^{1,2,3,5} = \frac{1}{4}\left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -2 \\
\hline
-1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 2 \\
1 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 2 \\
1 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -2 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & -2 & 2 & -2 & 0 & 0 & 2 & -2 & 0 & 0 & 4
\end{array}\right]$$

$$J_\mu^{1,2,3,5} = \frac{1}{4}\left[\begin{array}{ccc|ccc|ccc|ccc}
3 & 0 & 1 & -2 & 0 & -1 & 1 & 0 & 0 & -2 & 0 & 0 \\
0 & 2 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & -2 & 0 \\
1 & 0 & 3 & 0 & 0 & -1 & 1 & 0 & 2 & -2 & 0 & -4 \\
\hline
-2 & 1 & 0 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 3 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 2 & -1 & 1 & -1 & 2 & -2 & 0 \\
\hline
1 & -1 & 1 & -1 & 1 & -1 & 2 & 0 & 0 & -2 & 0 & 0 \\
0 & 1 & 0 & 0 & -2 & 1 & 0 & 3 & -1 & 0 & -2 & 0 \\
0 & -1 & 2 & 0 & 0 & -1 & 0 & -1 & 3 & 0 & 2 & -4 \\
\hline
-2 & 0 & -2 & 0 & 0 & 2 & -2 & 0 & 0 & 4 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & -2 & 0 & -2 & 2 & 0 & 4 & 0 \\
0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 8
\end{array}\right]$$

Note that $(1,2,3,4) \times (1,2,3,4)$ blocks of $J_\lambda^{1,2,3,5}$ and $J_\mu^{1,2,3,5}$ contribute to $(1,2,3,5) \times (1,2,3,5)$ blocks of J_λ and J_μ

Partial connection matrices of $\Diamond \mathrm{P}_3\mathrm{P}_4\mathrm{P}_1\mathrm{P}_5$ are as follows:

$$J_\lambda^{3,4,1,5} = \frac{1}{4}\left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & -1 & -1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\
\hline
-1 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\
1 & 0 & -1 & -1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\
-1 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & -2 & -2 & 2 & 0 & 0 & -2 & -2 & 0 & 0 & 4
\end{array}\right]$$

$$J_\mu^{3,4,1,5} = \frac{1}{4}\left[\begin{array}{ccc|ccc|ccc|ccc}
3 & 0 & -1 & -2 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\
0 & 2 & 0 & 1 & -1 & -1 & -1 & 1 & 1 & 0 & -2 & 0 \\
-1 & 0 & 3 & 0 & 0 & -1 & -1 & 0 & 2 & 2 & 0 & -4 \\
\hline
-2 & 1 & 0 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 3 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 2 & 1 & -1 & -1 & -2 & 2 & 0 \\
\hline
1 & -1 & -1 & -1 & 1 & 1 & 2 & 0 & 0 & -2 & 0 & 0 \\
0 & 1 & 0 & 0 & -2 & -1 & 0 & 3 & 1 & 0 & -2 & 0 \\
0 & 1 & 2 & 0 & 0 & -1 & 0 & 1 & 3 & 0 & -2 & -4 \\
\hline
-2 & 0 & 2 & 0 & 0 & -2 & -2 & 0 & 0 & 4 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 2 & 0 & -2 & -2 & 0 & 4 & 0 \\
0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 8
\end{array}\right]$$

Note that $(1,2,3,4) \times (1,2,3,4)$ blocks of $J_\lambda^{3,4,1,5}$ and $J_\mu^{3,4,1,5}$ contribute to $(3,4,1,5) \times (3,4,1,5)$ blocks of J_λ and J_μ

Synthesizing the above partial connection matrices yields the following connection matrices:

$$J_\lambda = \frac{1}{4}\left[\begin{array}{ccc|ccc|ccc|ccc|ccc}
1 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\
0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & -2 \\
1 & 1 & 2 & -1 & 1 & 0 & -1 & -1 & 2 & 1 & -1 & 0 & 0 & 0 & -4 \\
\hline
-1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & & & & 0 & 0 & 2 \\
1 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & & & & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & 0 & 0 & 0 \\
\hline
0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 2 \\
-1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \\
1 & 1 & 2 & -1 & 1 & 0 & -1 & -1 & 2 & 1 & -1 & 0 & 0 & 0 & -4 \\
\hline
0 & 1 & 1 & & & & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & -2 \\
0 & -1 & -1 & & & & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -2 & -4 & 2 & -2 & 0 & 2 & 2 & -4 & -2 & 2 & 0 & 0 & 0 & 8
\end{array}\right]$$

$$
J_\mu = \frac{1}{4}\left[\begin{array}{ccc|ccc|ccc|ccc|ccc}
5 & 0 & 1 & -2 & 0 & -1 & 2 & -1 & -1 & -1 & 1 & 1 & -4 & 0 & 0 \\
0 & 5 & 1 & 1 & -1 & 1 & -1 & 2 & -1 & 0 & -2 & -1 & 0 & -4 & 0 \\
1 & 1 & 6 & 0 & 0 & -1 & 1 & 1 & 4 & 0 & 0 & -1 & -2 & -2 & -8 \\
\hline
-2 & 1 & 0 & 3 & -1 & 0 & -1 & 0 & 0 & & & & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 3 & 0 & 1 & -2 & 0 & & & & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 2 & -1 & 1 & -1 & & & & 2 & -2 & 0 \\
\hline
2 & -1 & 1 & -1 & 1 & -1 & 5 & 0 & -1 & -2 & 0 & 1 & -4 & 0 & 0 \\
-1 & 2 & 1 & 0 & -2 & 1 & 0 & 5 & -1 & 1 & -1 & -1 & 0 & -4 & 0 \\
-1 & -1 & 4 & 0 & 0 & -1 & -1 & -1 & 6 & 0 & 0 & -1 & 2 & 2 & -8 \\
\hline
-1 & 0 & 0 & & & & -2 & 1 & 0 & 3 & -1 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & & & & 0 & -1 & 0 & -1 & 3 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & & & & 1 & -1 & -1 & 0 & 0 & 2 & -2 & 2 & 0 \\
\hline
-4 & 0 & -2 & 0 & 0 & 2 & -4 & 0 & 2 & 0 & 0 & -2 & 8 & 0 & 0 \\
0 & -4 & -2 & 0 & 0 & -2 & 0 & -4 & 2 & 0 & 0 & 2 & 0 & 8 & 0 \\
0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 16
\end{array}\right]
$$

Since either tetrahedron does not include both P_2 and P_4, $(2,4)$ and $(4,2)$ blocks of the connection matrices are zero matrices.

2.5 Implementation

Two-dimensional finite element calculation was implemented on MATLAB. Classes `Body`, `Triangle`, and `NodalPoint` were introduced. Class `Body` defines a two-dimensional body, which consists of an array of triangles and an array of nodal points. Class `Triangle` specifies a triangle, including three numbers of nodal points. Class `NodalPoint` defines a nodal point, including its two coordinates.

For example, rectangle region in Fig. 2.2 consists of 6 nodal points and 4 triangles. Coordinates of individual nodal points are listed as

$$
\text{points} = \left[\begin{array}{cccccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \boldsymbol{x}_4 & \boldsymbol{x}_5 & \boldsymbol{x}_6 \end{array}\right] = \left[\begin{array}{cccccc} 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]
$$

Nodal point numbers for individual triangular elements are listed as

$$
\text{triangles} = \left[\begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 5 & 4 & 2 \\ 6 & 5 & 3 \end{array}\right],
$$

which implies that $\triangle_1 = \triangle\mathrm{P}_1\mathrm{P}_2\mathrm{P}_4$, $\triangle_2 = \triangle\mathrm{P}_2\mathrm{P}_3\mathrm{P}_5$, $\triangle_3 = \triangle\mathrm{P}_5\mathrm{P}_4\mathrm{P}_2$, and $\triangle_4 = \triangle\mathrm{P}_6\mathrm{P}_5\mathrm{P}_3$. The rectangle region is then given by

```
elastic = Body(6, points, 4, triangles, thickness);
```

where `thickness` specifies thickness h of the two-dimensional body.

Instance of class `Triangle` includes geometric propertices such as nodal point numbers, area, and thickness as well as physical parameters such as density and Lamé's constants. Class `Triangle` involves the following methods:

partial_derivaties calculating partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$
calculate_Cauchy_strain calculating Cauchy strain in the triangle
partial_strain_potential_energy strain potential energy stored in the triangle

calculate_Green_strain calculating Green strain in the triangle
partial_strain_potential_energy_Green_strain strain energy using Green strain
partial_gravitational_potential_energy gravitational energy stored in the triangle
partial_stiffness_matrix calculating partial stiffness matrix $K_{i,j,k}$
partial_inertia_matrix calculating partial inertia matrix $M_{i,j,k}$
partial_gravitational_vector calculating partial gravitational vector $\boldsymbol{g}_{i,j,k}$

Class `Body` involves the following methods:

total_strain_potential_energy calculating strain energy stored in the body
total_strain_potential_energy_Green_strain strain energy using Green strain
total_gravitational_potential_energy gravitational energy stored in the body
calculate_stiffness_matrix calculating stiffness matrix K
calculate_inertia_matrix calculating inertia matrix M
calculate_gravitational_vector calculating gravitational vector $\boldsymbol{g}$
constraint_matrix constraint matrix when specified nodal points are fixed
draw draw the shape of the body

Assuming that density ρ and Lamé's constants λ, μ are uniform over the region, the following specifies these parameters:

```
elastic = elastic.mechanical_parameters(rho, lambda, mu);
```

The following calculates the stiffness and inertia matrices:

```
elastic = elastic.calculate_stiffness_matrix;
elastic = elastic.calculate_inertia_matrix;
```

The stiffness and inertia matrices are then referred by

```
M = elastic.Inertia_Matrix;
K = elastic.Stiffness_Matrix;
```

which can be applied to static and dynamic calculation of the motion and deformation of a soft body.

Three-dimensional finite element calculation was implemented on MATLAB. Classes `Body`, `Tetrahedron`, and `NodalPoint` were introduced. Class `Body` defines a three-dimensional body, which consists of an array of tetrahedra and an array of nodal points. Class `Tetrahedron` specifies a tetrahedron, including four numbers of nodal points. Class `NodalPoint` defines a nodal point, including its three coordinates.

For example, a regular quadrangular pyramid (Fig. 2.3) consists of 5 nodal points and 2 tetrahedra. Coordinates of individual nodal points are listed as

$$\text{points} = \left[\begin{array}{ccccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \boldsymbol{x}_4 & \boldsymbol{x}_5 \end{array} \right] = \left[\begin{array}{ccccc} 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

Nodal point numbers for individual tetrehedron elements are listed as

$$\text{tetrahedra} = \left[\begin{array}{cccc} 1 & 2 & 3 & 5 \\ 3 & 4 & 1 & 5 \end{array} \right],$$

which implies that $\lozenge_1 = \lozenge \mathrm{P}_1\mathrm{P}_2\mathrm{P}_3\mathrm{P}_5$ and $\lozenge_2 = \lozenge \mathrm{P}_3\mathrm{P}_4\mathrm{P}_1\mathrm{P}_5$. The quadrangular pyramid is then given by

```
elastic = Body(5, points, 2, tetrahedra);
```

followed by methods to define physical parameters and calculate inertia and stiffness matrices.

Problems

1. Show eqs. (2.1.3a)(2.1.3b)(2.1.3c).

2. Calculate partial derivatives of piecewise linear approximation $L_{i,j,k}(x, y)$ given in eq. (2.1.4) with respect to x, y.

3. Calculate partial derivatives of piecewise linear approximation $L_{i,j,k,l}(x, y, z)$ given in eq. (2.1.6) with respect to x, y, z.

4. Show the following equations:

$$\int_{x_i}^{x_j} N_{i,j}(x)\, N_{i,j}(x)\, \mathrm{d}x = \int_{x_i}^{x_j} N_{j,i}(x)\, N_{j,i}(x)\, \mathrm{d}x = \frac{1}{3}(x_j - x_i)$$
$$\int_{x_i}^{x_j} N_{i,j}(x)\, N_{j,i}(x)\, \mathrm{d}x = \int_{x_i}^{x_j} N_{j,i}(x)\, N_{i,j}(x)\, \mathrm{d}x = \frac{1}{6}(x_j - x_i)$$

   Letting $L_{i,j}(x) = f_i\, N_{i,j}(x) + f_j\, N_{j,i}(x)$, show

$$\int_{x_i}^{x_j} \{L_{i,j}(x)\}^2\, \mathrm{d}x = \begin{bmatrix} f_i & f_j \end{bmatrix} \frac{x_j - x_i}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} f_i \\ f_j \end{bmatrix}$$

5. Show the following equations:

$$\int_{x_i}^{x_j} N'_{i,j}(x)\, N'_{i,j}(x)\, \mathrm{d}x = \int_{x_i}^{x_j} N'_{j,i}(x)\, N'_{j,i}(x)\, \mathrm{d}x = \frac{1}{x_j - x_i}$$
$$\int_{x_i}^{x_j} N'_{i,j}(x)\, N'_{j,i}(x)\, \mathrm{d}x = \int_{x_i}^{x_j} N'_{j,i}(x)\, N'_{i,j}(x)\, \mathrm{d}x = \frac{-1}{x_j - x_i}$$

   Letting $L_{i,j}(x) = f_i\, N_{i,j}(x) + f_j\, N_{j,i}(x)$, show

$$\int_{x_i}^{x_j} \left\{L'_{i,j}(x)\right\}^2\, \mathrm{d}x = \begin{bmatrix} f_i & f_j \end{bmatrix} \frac{1}{x_j - x_i} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} f_i \\ f_j \end{bmatrix}$$

6. Show the following equations:

$$\int_{\triangle} N_{i,j,k}^2\, \mathrm{d}S = \int_{\triangle} N_{j,k,i}^2\, \mathrm{d}S = \int_{\triangle} N_{k,i,j}^2\, \mathrm{d}S = \frac{\triangle}{6}$$
$$\int_{\triangle} N_{i,j,k}\, N_{j,k,i}\, \mathrm{d}S = \int_{\triangle} N_{j,k,i}\, N_{k,i,j}\, \mathrm{d}S = \int_{\triangle} N_{k,i,j}\, N_{i,j,k}\, \mathrm{d}S = \frac{\triangle}{12}$$

   where $\triangle = \triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$.

7. Show eq. (2.3.13).

8. Show the following equations:

$$\int_{\Diamond} N_{i,j,k,l}^2\, \mathrm{d}V = \cdots = \frac{\Diamond}{10}$$
$$\int_{\Diamond} N_{i,j,k,l} N_{j,k,l,i}\, \mathrm{d}V = \cdots = \frac{\Diamond}{20}$$

   where $\Diamond = \Diamond \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l$.