Chapter 3

Computing Static Deformation

3.1 Variational principle in statics

Let us calculate the static deformation of an elastic body. We apply variational principle in statics for the calculation. Let U be potential energy of the body. External forces applied to the body will deform the body. Let W be work done by external forces. Geometric constraints imposed on the body causes the deformation of the body. Let $\boldsymbol{R}$ be a collective vector of geometric constraints. Variational principle in statics insists that internal energy $I = U - W$ reaches to its minimum under geometric constraints $\boldsymbol{R} = \boldsymbol{0}$ at its equilibrium. In other word, we can calculate the static deformation of the body by minimizing the internal energy under geometric constraints:

$$\begin{aligned} &\text{minimize} \;\; I = U - W \\ &\text{subject to} \;\; \boldsymbol{R} = \boldsymbol{0} \end{aligned} \tag{3.1.1}$$

In finite element approximation, deformation of an elastic body is described by nodal displacement vector $\boldsymbol{u}_{\mathrm{N}}$, implying that internal energy and geometric constraints are functions of vector $\boldsymbol{u}_{\mathrm{N}}$. Calculating vector $\boldsymbol{u}_{\mathrm{N}}$ that minimizes internal energy $I(\boldsymbol{u}_{\mathrm{N}})$ under geometric constraints $\boldsymbol{R}(\boldsymbol{u}_{\mathrm{N}}) = \boldsymbol{0}$ yields static deformation of the body.

The above conditional minimization problem can be solved numerically or analytically. One method is the direct application of numerical optimization algorithms. Many optimization algorithms have been proposed and available. We can apply such optimization algorithms to the above problem. For example, MATLAB offers function `fmincon` for conditional minimization, with `optimization` toolbox. Applying function `fmincon` to the above problem yields vector $\boldsymbol{u}_{\mathrm{N}}$, which describes the static deformation of the body. Another method is analytical. The above conditional minimization problem can be converted into unconditional problem as

$$I' = I - \boldsymbol{\lambda}^{\top}\boldsymbol{R} = U - W - \boldsymbol{\lambda}^{\top}\boldsymbol{R}$$

where $\boldsymbol{\lambda}$ denote a collective vector consisting of Lagrange multipliers corresponding to individual constraints. The above function is stationary at the static deformation, resulting

$$\frac{\partial I'}{\partial \boldsymbol{u}_{\mathrm{N}}} = \boldsymbol{0}$$

Thus, solving $\partial I'/\partial \boldsymbol{u}_{\mathrm{N}} = \boldsymbol{0}$ with $\boldsymbol{R} = \boldsymbol{0}$, we can compute static deformation. When both equations are linear, we can solve the combined linear equation analytically or numerically, yielding the static deformation of the body.

3.2 Static deformation of one-dimensional soft body

Let us calculate the static deformation of a regular-shaped elastic beam. Assume that cross-sectional area A and Young's modulus E are constant. Dividing $[\,0,\ L\,]$ into four small regions, strain potential energy of the beam is described by a quadratic form with respect to nodal displacement vector:

$$U = \frac{1}{2}\,\boldsymbol{u}_{\rm N}^{\top}\,K\,\boldsymbol{u}_{\rm N}$$

(see eq. (2.2.6)), where stiffness matrix K is given in eq. (2.2.5). Assume that end point P(0) is fixed to space while an external force f is applied to end point P(L). Work done by the external force is then described as

$$W = \boldsymbol{f}^{\top}\boldsymbol{u}_{\rm N}$$

where $\boldsymbol{f} = [\,0,\ 0,\ 0,\ 0,\ f\,]^{\top}$. Since displacement of point P(0) should be equal to zero, the following geometric constraint must be satisfied:

$$R = \boldsymbol{a}^{\top}\boldsymbol{u}_{\rm N} = 0$$

where $\boldsymbol{a} = [\,1,\ 0,\ 0,\ 0,\ 0\,]^{\top}$. Consequently, minimization problem to compute static deformation turns into:

$$\begin{aligned} &\text{minimize} \quad I = \frac{1}{2}\,\boldsymbol{u}_{\rm N}^{\top}\,K\,\boldsymbol{u}_{\rm N} - \boldsymbol{f}^{\top}\boldsymbol{u}_{\rm N} \\ &\text{subject to} \quad \boldsymbol{a}^{\top}\boldsymbol{u}_{\rm N} = 0 \end{aligned} \tag{3.2.1}$$

Let us apply analytical method to solve the above problem. Note that

$$I' = \frac{1}{2}\,\boldsymbol{u}_{\rm N}^{\top}\,K\,\boldsymbol{u}_{\rm N} - \boldsymbol{f}^{\top}\boldsymbol{u}_{\rm N} - \lambda_a \boldsymbol{a}^{\top}\boldsymbol{u}_{\rm N},$$

where λ_a is a Lagrange multiplier corresponding to a single constraint $\boldsymbol{a}^{\top}\boldsymbol{u}_{\rm N} = 0$, is stationary at the minimum under the constraint. Since matrix K and vector $\boldsymbol{f}$ are constant, we have

$$\frac{\partial I'}{\partial \boldsymbol{u}_{\rm N}} = K\boldsymbol{u}_{\rm N} - \boldsymbol{f} - \lambda_a \boldsymbol{a} = \boldsymbol{0}$$

which directly yields

$$\begin{bmatrix} K & -\boldsymbol{a} \\ -\boldsymbol{a}^{\top} & \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\rm N} \\ \lambda_a \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix}. \tag{3.2.2}$$

Solving the above linear equation, we obtain nodal displacement vector $\boldsymbol{u}_{\rm N}$, which describes the static deformation of the beam. Additionally, Lagrange multiplier λ_a represents a constraint force corresponding to $\boldsymbol{a}^{\top}\boldsymbol{u}_{\rm N} = 0$, that is the reaction force at the fixed point P(0). Solution of the above linear equation is given by $\boldsymbol{u}_{\rm N} = f/(EA/h)\,[\,0,\ 1,\ 2,\ 3,\ 4\,]^{\top}$ and $\lambda_a = -f$, implying that the beam extends uniformly.

Irregular-shaped beam Let us calculate the static deformation of an irregular-shaped beam. Assume that cross-section area of the beam depends on x, given by $A(x) = a - 2bx$, where a, b are positive constants satisfying $A(L) > 0$. Dividing $[\,0,\ L\,]$ into four small regions, stiffness matrix of the beam is described as eq. (2.2.8). We apply numerical intergal to calculate $V_{i,j}$ given in eq. (2.2.7). Letting $E = 2$, $\rho = 1.0$, $L = 10$, $a = 4$, and $b = 0.1$, the stiffness matrix of the beam is calculated as

$$K = \begin{bmatrix} 3.0000 & -3.0000 & & & \\ -3.0000 & 5.6000 & -2.6000 & & \\ & -2.6000 & 4.8000 & -2.2000 & \\ & & -2.2000 & 4.0000 & -1.8000 \\ & & & -1.8000 & 1.8000 \end{bmatrix}$$

Solving eq. (3.2.2), we have $\boldsymbol{u}_\mathrm{N} = f\,[\,0.0000,\ 0.3333,\ 0.7179,\ 1.1725,\ 1.7280\,]^\top$ and $\lambda_a = -f$, implying that deformation is not uniform; beam top near $\mathrm{P}(L)$ extends more.

Fixing both ends Assume that both ends of a beam are fixed to space and external force f is applied to the center of the beam. Divide $[\,0,\ L\,]$ into four small regions. Work done by the external force is then described as $W = \boldsymbol{f}^\top \boldsymbol{u}_\mathrm{N}$, where $\boldsymbol{f} = [\,0,\ 0,\ f,\ 0,\ 0\,]^\top$. Here we have two geometric constraints:

$$R_1 = \boldsymbol{a}_1^\top \boldsymbol{u}_\mathrm{N} = 0, \qquad R_2 = \boldsymbol{a}_2^\top \boldsymbol{u}_\mathrm{N} = 0$$

where $\boldsymbol{a}_1 = [\,1,\ 0,\ 0,\ 0,\ 0\,]^\top$ and $\boldsymbol{a}_2 = [\,0,\ 0,\ 0,\ 0,\ 1\,]^\top$. Then, we find

$$I' = \frac{1}{2}\,\boldsymbol{u}_\mathrm{N}^\top\, K\, \boldsymbol{u}_\mathrm{N} - \boldsymbol{f}^\top \boldsymbol{u}_\mathrm{N} - \lambda_1 \boldsymbol{a}_1^\top \boldsymbol{u}_\mathrm{N} - \lambda_2 \boldsymbol{a}_2^\top \boldsymbol{u}_\mathrm{N}$$

where λ_1 and λ_2 are Langrange multipliers corresponding to the two constraints. Introducing a collective vector $\boldsymbol{\lambda} = [\,\lambda_1,\ \lambda_2\,]^\top$ and matrix $A = [\,\boldsymbol{a}_1,\ \boldsymbol{a}_2\,]$, the above equation turns into:

$$I' = \frac{1}{2}\,\boldsymbol{u}_\mathrm{N}^\top\, K\, \boldsymbol{u}_\mathrm{N} - \boldsymbol{f}^\top \boldsymbol{u}_\mathrm{N} - (A\boldsymbol{\lambda})^\top \boldsymbol{u}_\mathrm{N}.$$

Two constraints are collectively given by $A^\top \boldsymbol{u}_\mathrm{N} = \boldsymbol{0}$. Linear equation eq. (3.2.2) to compute static deformation then turns into:

$$\begin{bmatrix} K & -A \\ -A^\top & \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_\mathrm{N} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}. \tag{3.2.3}$$

The above equation provides a general description for computing static deformation.

3.3 Static deformation of two-dimensional soft body

Let us calculate the static deformation of an elastic rectangle region shown in Fig. 2.2. Assume that λ and μ are uniform over the region. Elastic potential energy of the rectangle region is described by a quadratic form with respect to nodal displacement vector:

$$U = \frac{1}{2}\,\boldsymbol{u}_\mathrm{N}^\top\, K\, \boldsymbol{u}_\mathrm{N}$$

(see eq. (2.3.17)), where stiffness matrix K is given in eq. (2.3.19). Assume that edge $\mathrm{P}_1\mathrm{P}_4$ is fixed to a wall and a constant external force $\boldsymbol{f}_\mathrm{ext} = [\,f_x,\ f_y\,]^\top$ is applied to the center point of edge $\mathrm{P}_3\mathrm{P}_6$. Since displacement of the center point is given by $(\boldsymbol{u}_3 + \boldsymbol{u}_6)/2$, work done by external force is formulated as

$$W = \boldsymbol{f}_\mathrm{ext}^\top \left(\frac{\boldsymbol{u}_3 + \boldsymbol{u}_6}{2} \right) = \boldsymbol{f}^\top \boldsymbol{u}_\mathrm{N}$$

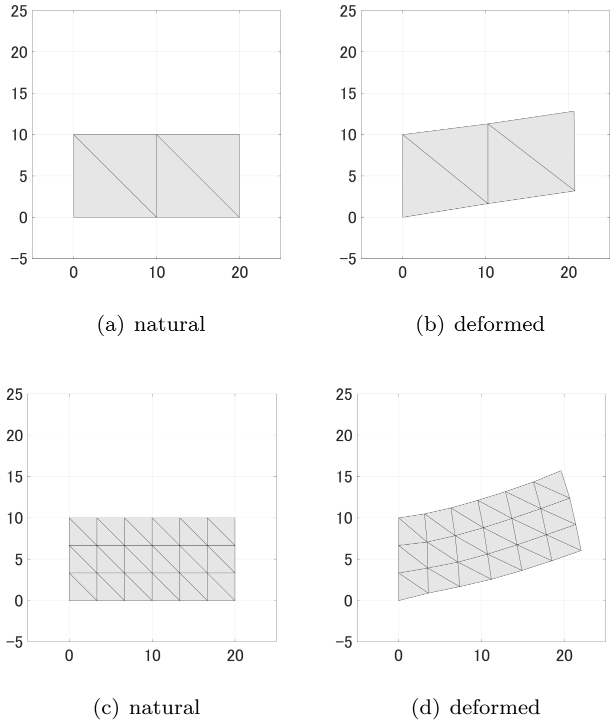


Figure 3.1: Calculated deformation of a rectangular elastic body

where

$$\boldsymbol{f} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{f}_{\rm ext}/2 \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{f}_{\rm ext}/2 \end{bmatrix}.$$

Since two points P_1 and P_4 are fixed to a wall, we find that $\boldsymbol{u}_1 = \mathbf{0}$ and $\boldsymbol{u}_4 = \mathbf{0}$ must be satisfied. Then, a set of constraints can be described as

$$\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_4 \end{bmatrix} = A^\top \boldsymbol{u}_{\rm N} = \mathbf{0}_4$$

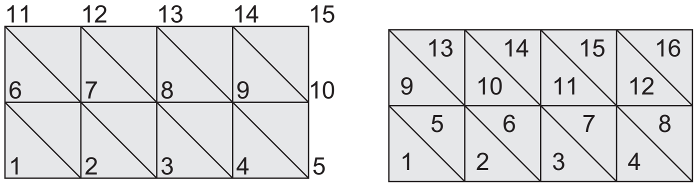
where

$$A^\top = \begin{bmatrix} I_{2\times2} & O_{2\times2} & O_{2\times2} & O_{2\times2} & O_{2\times2} & O_{2\times2} \\ O_{2\times2} & O_{2\times2} & O_{2\times2} & I_{2\times2} & O_{2\times2} & O_{2\times2} \end{bmatrix}.$$

Then, we obtain linear equation given in eq. (3.2.3). Solving the linear equation yields static deformation of the rectangle region.

Figure 3.1 shows calculation results for Young's modulus $E = 0.1\,\mathrm{MPa}$, Poisson's ratio $\nu = 0.48$, width 20 cm, height 10 cm, thickness $h = 1\,\mathrm{cm}$, and external force $\boldsymbol{f}_{\rm ext} = [\,10,\ 5\,]^\top\,\mathrm{N}$. The body consists of $2 \times 1 \times 2$ triangles in Figs. 3.1(a) and 3.1(b) while $6 \times 3 \times 2$ triangles in Figs. 3.1(c) and 3.1(d). A finer mesh can describe more detailed deformation.

Numbering in nodal points and triangles Let us number nodal points and triangles of a rectangular body. Figure 3.2 shows a rectangular body, consisting of 15 nodal points and 16 triangles. Consecutive numbers are assigned to nodal points / triangles from left-bottom to right-top. Triangle T_1 consists of three nodal points P_1, P_2, and P_6, that is, $T_1 = \triangle P_1P_2P_6$. Triangle T_{16} consists of three nodal points P_{15}, P_{14}, and P_{10}, that is, $T_{16} = \triangle P_{15}P_{14}P_{10}$. This numbering will be applied to two-dimensional rectangular regions.



(a) nodal point numbers (b) triangle numbers

Figure 3.2: Rectangular body

Example (push test) Let us push an elastic square body on a table as shown in Fig. 3.3. A rigid rectangular plate will push the top surface of the body downward. Assume that the plate is parallel to the floor during its motion and the top surface is fixed to the plate. External force f_{push} is applied to push the plate downward. Divide the square region into $4 \times 4 \times 2$ triangles as shown in the figure. Deformation of the elastic body is described by displacement vectors of nodal points P_1 through P_{25}. Nodal points P_1 through P_5 are contacting to the floor and P_{21} through P_{25} are contacting to the plate. Let d_{push} be the pushed distance of the plate. Then, constraints are formulated as follows:

$$\begin{aligned} &\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}_3 = \boldsymbol{u}_4 = \boldsymbol{u}_5 = \boldsymbol{0}, \\ &\boldsymbol{u}_{21} = \boldsymbol{u}_{22} = \boldsymbol{u}_{23} = \boldsymbol{u}_{24} = \boldsymbol{u}_{25} = -d_{\text{push}}\boldsymbol{e}_y \end{aligned}$$

where $\boldsymbol{e}_y = [\,0,\,1\,]^{\top}$. These constraints are then collectively described as follows:

$$A^{\top}\boldsymbol{u}_{\text{N}} + \boldsymbol{d}d_{\text{push}} = \boldsymbol{0}_{20}$$

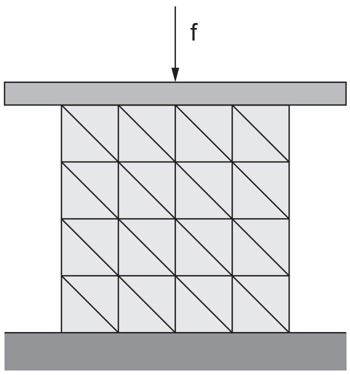
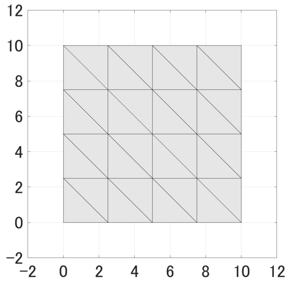
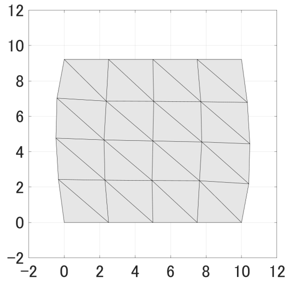


Figure 3.3: Push test of an elastic body



(a) natural



(b) deformed

Figure 3.4: Deformation in push test

where

$$A^{\top} = \left[\begin{array}{c|c|c} I_{10\times 10} & O_{10\times 30} & O_{10\times 10} \\ \hline O_{10\times 10} & O_{10\times 30} & I_{10\times 10} \end{array} \right], \quad \boldsymbol{d} = \left[\begin{array}{c} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hline \boldsymbol{e}_y \\ \vdots \\ \boldsymbol{e}_y \end{array} \right]$$

Work done by external force f is described as $W = f_{\text{push}} d_{\text{push}}$. Consequently, we have the following conditional minimization problem:

$$\begin{aligned} &\text{minimize} \quad I = \frac{1}{2}\boldsymbol{u}^{\top} K \boldsymbol{u} - f_{\text{push}} d_{\text{push}} \\ &\text{subject to} \quad A^{\top}\boldsymbol{u}_{\text{N}} + \boldsymbol{d} d_{\text{push}} = \mathbf{0}_{20} \end{aligned}$$

where K denotes the stiffness matrix. Finally, we have the following equation:

$$\begin{bmatrix} K & \mathbf{0}_{50} & -A \\ \mathbf{0}_{50}^{\top} & 0 & -\boldsymbol{d}^{\top} \\ -A^{\top} & -\boldsymbol{d} & O_{20\times 20} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\text{N}} \\ d_{\text{push}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \\ \mathbf{0}_{20} \end{bmatrix}$$

where $\boldsymbol{\lambda}$ consists of Lagrange multipliers corresponding to the constraints. Solving the above equation, we obtain the displacement of the plate, that is, d_{push} for given pushing force f_{push}. Figure 3.4 shows a computation result. An elastic body of width 10 cm, thickness $h = 1$ cm, Young's modulus $E = 0.1$ MPa, and Poisson's ratio $\nu = 0.48$ on a floor deforms according to pushing force 20 N. The deformed shape is almost symmetric, but not completely symmetric due to an asymmetric mesh.

Applying conditional optimization algorithm We can solve conditional minimization problem eq. (3.1.1) directly using a conditional optimization algorithm.

Potential energy is a function of nodal displacement vector $\boldsymbol{u}_{\text{N}}$. Let us calculate strain potential energy U_p stored in triangle $\text{T}_p = \triangle \text{P}_i \text{P}_j \text{P}_k$. Procedure to calculate U_p is summarized as follows:

function U_p = strain_potential_energy_in_triangle ($\boldsymbol{u}_\mathrm{N}$)
Obtain displacements $\boldsymbol{u}_i$, $\boldsymbol{u}_j$, $\boldsymbol{u}_k$.
Calculate vectors $\boldsymbol{a}$, $\boldsymbol{b}$ (eq. (2.3.11)) and partial derivatives u_x, u_y, v_x, v_y
Calculate strain vector $\boldsymbol{\varepsilon}$ (eq. (2.3.12)).
Calculate potential energy $U_p = U_{i,j,k}$ stored in triangle T_p (eq. (2.3.15)).

Total strain potential energy is given by summing up potential energies U_p as

$$U = \sum_p U_p. \tag{3.3.1}$$

A function to calculate the internal energy is summarized as

function I = internal_energy ($\boldsymbol{u}_\mathrm{N}$)
Calculate potential energies U_p for all triangles.
Sum up all potential energies U_p to obtain total potential energy U.
Calcu;ate work W done by external forces.
Calculate $I = U - W$.

Conditional minimization problem eq. (3.1.1) then described as

$$\begin{aligned} &\text{minimize} \quad I(\boldsymbol{u}_\mathrm{N}) \\ &\text{subject to} \quad A^\top \boldsymbol{u}_\mathrm{N} = \boldsymbol{0} \end{aligned} \tag{3.3.2}$$

which can be solved numerically using a conditional optimization algorithm. For example, we apply function `fmincond` in MATLAB, which minimizes an objective function under linear equations, linear inequalities, and nonlinear constraints.

Example (deformation by pressure) Let us calculate the deformation of an elastic membrane deformed by air pressure. An elastic membrane is attached to a rigid shell (Fig. 3.5). Pressure p is applied into a chamber surrounded by the membrane and the shell. The applied pressure deforms the membrane. Work done by constant pressure p is formulated as

$$W = pV \tag{3.3.3}$$

where V denote the increment of the volume of the chamber. In two-dimensional deformation, V is given by hS, where S denote the increment of the area of the chamber.

Let us attach an elastic membrane of its width 10 cm, height 1 cm, thickness $h = 1$ cm, Young's modulus $E = 0.1$ MPa, and Poisson's ratio $\nu = 0.48$ to the rigid shell. Divide the membrane region into $10 \times 1 \times 2$ triangles (Fig. 3.6(a)). Pressure is applied to bottom edges $\mathrm{P}_1\mathrm{P}_2$ through $\mathrm{P}_{10}\mathrm{P}_{11}$. The increment of the chamber area is specified by polygon $\mathrm{P}_{11}\mathrm{P}_{10}\cdots\mathrm{P}_2\mathrm{P}_1$. The signed polygon area $S(\boldsymbol{u}_\mathrm{N})$ takes positive value when the membrane expands while negative value when it shrinks. Method **surrounded_area** of class `Body` calculates the area of deformed polygon, given a list of nodal point numbers of the polygon and displacements of the nodal points, which describe the deformation of the polygon (see Problem 6). Finally, we have the following conditional optimization problem:

$$\begin{aligned} &\text{minimize} \quad I(\boldsymbol{u}_\mathrm{N}) = \frac{1}{2}\boldsymbol{u}_\mathrm{N}^\top K \,\boldsymbol{u}_\mathrm{N} - ph\, S(\boldsymbol{u}_\mathrm{N}) \\ &\text{subject to} \quad A^\top \boldsymbol{u}_\mathrm{N} = \boldsymbol{0} \end{aligned} \tag{3.3.4}$$

Note that $A^\top \boldsymbol{u}_\mathrm{N} = \boldsymbol{0}$ corresponds to a set of geometric constraints; $\boldsymbol{u}_1 = \boldsymbol{0}$, $\boldsymbol{u}_{11} = \boldsymbol{0}$, $\boldsymbol{u}_{12} = \boldsymbol{0}$, and $\boldsymbol{u}_{22} = \boldsymbol{0}$. Applying a conditional optimization algorithm to the above problem, we

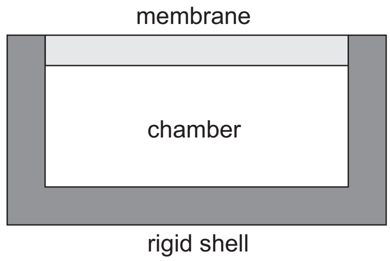


Figure 3.5: Elastic membrane deformed by air pressure

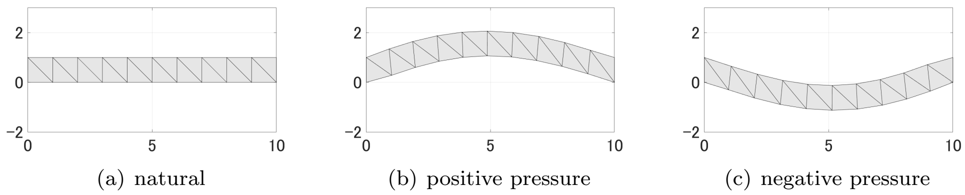


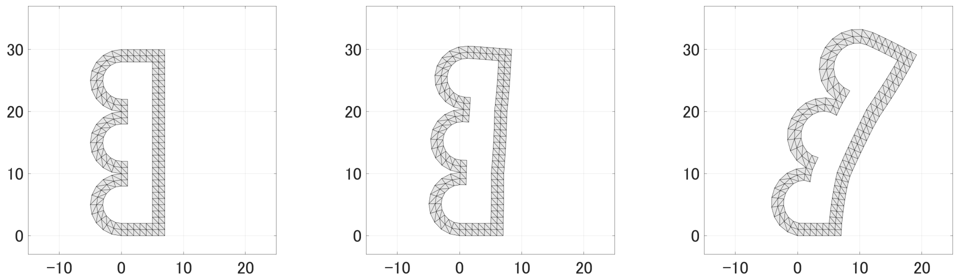
Figure 3.6: Deformation of membrane by pressure

obtain nodal displacement vector $\boldsymbol{u}_{\mathrm{N}}$, which describes the deformed shape of the membrane. Figure 3.6(b) shows a computation result at applied pressure $p = 2\,\mathrm{kPa}$. The membrane deforms outward. As $S(\boldsymbol{u}_{\mathrm{N}})$ denotes the signed area, which could take a negative value, we can calculate the deformation corresponding to negative pressure. Figure 3.6(c) shows a computation result at negative pressure $p = -2\,\mathrm{kPa}$. The membrane deforms inward, which describes the actual phenomena.

Example (deformation of a PneuNet actuator) Let us calculate the static deformation of a PneuNet actuator (Fig. 3.7(a)). This actuator is composed of elastic material of Young's modulus $E = 0.1\,\mathrm{MPa}$ and Poisson's ratio $\nu = 0.48$, and involves a series of three air chambers along its left side. Air pressure is applied inside the actuator, expanding the air chambers, which yields the bend deformation of the actuator (Figs. 3.7(b) and 3.7(c)). Nodal points on the bottom side are fixed to the floor. Actual PneuNet actuators are three-dimensional; junctions between neighboring air chambers and right side of the actuator are connected by front and back elastic covers, resulting that distance between a junction and the right side remains almost constant. So, in this calculation, we impose two additional geometric constraints that the distances between individual junctions and the right side remain constant. The two junction are specified by P_{100} and P_{105}, their corresponding nodal points are given by P_{153} and P_{183}, respectively. Consequently, we have the following conditional optimization problem:

$$
\begin{aligned}
&\text{minimize} && I(\boldsymbol{u}_{\mathrm{N}}) \\
&\text{subject to} && A^{\top}\boldsymbol{u}_{\mathrm{N}} = \boldsymbol{0} \\
& && R_1 = \| \boldsymbol{r}_{100} - \boldsymbol{r}_{153} \| - \| \boldsymbol{x}_{100} - \boldsymbol{x}_{153} \| = 0 \\
& && R_2 = \| \boldsymbol{r}_{105} - \boldsymbol{r}_{183} \| - \| \boldsymbol{x}_{105} - \boldsymbol{x}_{183} \| = 0
\end{aligned}
\tag{3.3.5}
$$

where matrix A originates from constraints imposed on nodal points on the bottom side and $\boldsymbol{r}_k = \boldsymbol{x}_k + \boldsymbol{u}_k$ where $k = 100$, 153, 105, and 183. Figure 3.7(b) shows the deformation at applied pressure of $500\,\mathrm{kPa}$ and Fig. 3.7(c) shows the deformation at applied pressure of



(a) natural (b) 500 kPa (c) 700 kPa

Figure 3.7: Deformation of PneuNet actuator

700 kPa. It turns out that applying 500 kPa pressure causes little deformation but 700 kPa pressure yields much deformation. This computations was performed by MATLAB running on Windows 10, i5–6300U CPU at 2.40 GHz with 8.0 GB memory. Computation time was about 55 minutes.

Let us impose equilibrium equation to speed up the above computation. Let $\boldsymbol{\lambda}$ is a set of Lagrange multipliers corresponding to $A^{\top}\boldsymbol{u}_{\rm N} = \boldsymbol{0}$, λ_1 and λ_2 are Lagrange multipliers corresponding to $R_1 = 0$ and $R_2 = 0$. The equilibrium equation is then described as follows:

$$K\boldsymbol{u}_{\rm N} - \boldsymbol{f}_p - A\boldsymbol{\lambda} - \boldsymbol{g}_1\lambda_1 - \boldsymbol{g}_2\lambda_2 = \boldsymbol{0}, \tag{3.3.6}$$

where K is the stiffness matrix, $\boldsymbol{f}_p = ph\, \partial S/\partial \boldsymbol{u}_{\rm N}$ denotes a set of nodal forces caused by pressure p, $\boldsymbol{g}_1 = \partial R_1/\partial \boldsymbol{u}_{\rm N}$, and $\boldsymbol{g}_2 = \partial R_2/\partial \boldsymbol{u}_{\rm N}$. Since Lagrange multipliers are additional unknowns, we have the following conditional optimization problem:

$$\begin{aligned}
&\text{minimize} && I(\boldsymbol{u}_{\rm N}) \\
&\text{subject to} && A^{\top}\boldsymbol{u}_{\rm N} = \boldsymbol{0} \\
& && R_1 = \| \boldsymbol{r}_{100} - \boldsymbol{r}_{153} \| - \| \boldsymbol{x}_{100} - \boldsymbol{x}_{153} \| = 0 \\
& && R_2 = \| \boldsymbol{r}_{105} - \boldsymbol{r}_{183} \| - \| \boldsymbol{x}_{105} - \boldsymbol{x}_{183} \| = 0 \\
& && \begin{bmatrix} K & -A & -\boldsymbol{g}_1 & -\boldsymbol{g}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\rm N} \\ \boldsymbol{\lambda} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \boldsymbol{f}_p
\end{aligned} \tag{3.3.7}$$

The deformed shape of a PneuNet actuator can be calculated by solving the above conditional optimization problem numerically. We obtained the deformed shapes shown in Fig. 3.7. Computation time was less than 50 s, implying that imposing equilibrium equation realizes over 60 times speed up in calculation.

3.4 Inhomogeneous elasticity

So far, we have assumed that mechanical properties are homogeneous. When Lamé's constants are uniform over an elastic body, its stiffness matrix is formulated in eq. (2.3.19). Here we focus on inhomogeneous properties. Let us divide a two-dimensional elastic body into a finite number of triangles. Let K_p be partial stiffness matrix of triangle $\mathrm{T}_p = \triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k$. Assume that Lamé's constants are uniform over triangle T_p, which are denoted as λ_p and

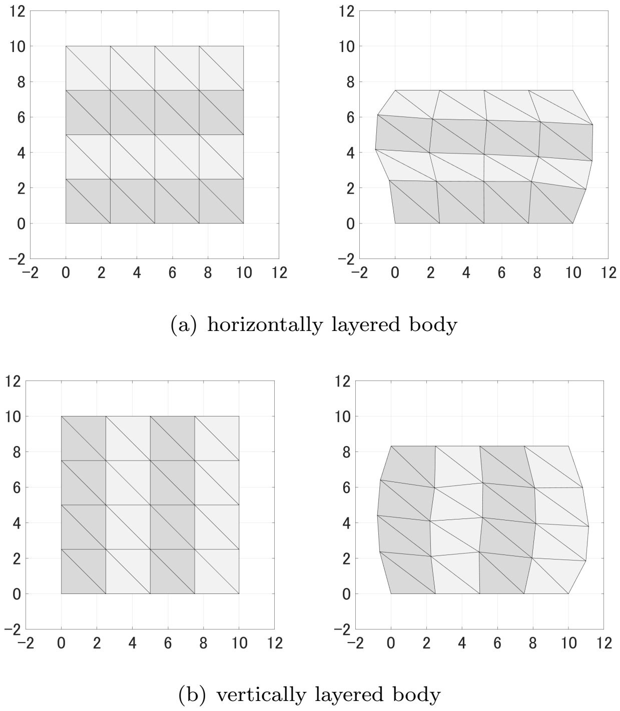


Figure 3.8: Deformation of layered bodies in push test

μ_p. Namely, individual triangles may have different values of Lamé's constants. The partial stiffness matrix is then described as

$$K_p = \lambda_p J_\lambda^p + \mu_p J_\mu^p$$

where $J_\lambda^p = J_\lambda^{i,j,k}$ and $J_\mu^p = J_\mu^{i,j,k}$ be partial connection matrices of triangle T_p. Synthesizing partial stiffness matrices of all triangles, we obtain the total stiffness matrix:

$$K = \bigoplus_p K_p$$

Note that this equation is equivalent to eq. (2.3.18).

Let us apply push test to a horizontally layered body (Fig. 3.8(a) left) and a vertically layered body (Fig. 3.8(b) left). These layered bodies of width 10 cm and thickness $h = 1$ cm consist of two materials. Dark region corresponds to $E = 0.1$ MPa and $\nu = 0.48$ while light region corresponds to $E = 0.01$ MPa and $\nu = 0.48$. Namely, material in dark region is ten-times harder than material in light region. Bottom surface is fixed to the floor and top surface is push downward by a rigid rectangular plate. Deformed shapes at pushing force 20 N are shown in Figs. 3.8(a) and 3.8(b). Deformed shapes are different from the deformed shape of a uniform elastic body (Fig. 3.3). Additionally, it turns out that a horizontally layered body deforms (Fig. 3.8(a) right) more than a vertically layered body deforms (Fig. 3.8(b) right). Namely, a horizontally layered body is softer that a vertically layered body. Figure 3.9 shows force-displacement relationship in push test. Four bodies, a horizontally layered body, a vertically layered body, a body consisting of hard material alone, and a body consisting of soft material alone. This also describes that a horizontally layered body is softer that a vertically layered body.

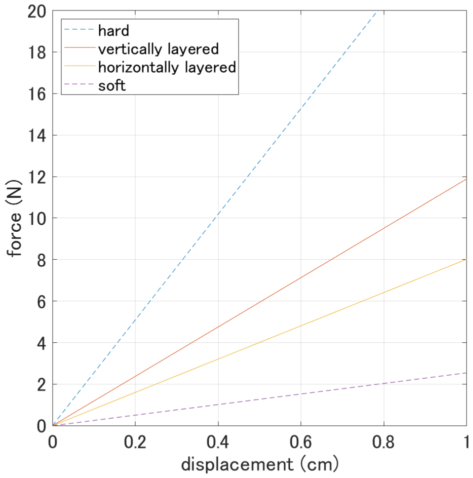


Figure 3.9: Force-displacement relationship in push test

Class `SubRegion` was introduced to define a region consisting of triangles. For example, hard region in a horizontally layered body (Fig. 3.8(a)) consists of triangles T_1 through T_8 and T_{17} through T_{24}, which is given as

```
subregion_hard = [ 1:8, 17:24 ];
elastic = elastic.define_subregion(subregion_hard);
elastic = elastic.subregion_mechanical_parameters(density, lambda, mu);
```

followed by specification of mechanical parameters over the hard region.

3.5 Green strain

Strain vector $\boldsymbol{E} = [\, E_{xx},\, E_{yy},\, 2E_{xy}\,]^\top$, where

$$E_{xx} = u_x + \frac{1}{2}(u_x^2 + v_x^2) \tag{3.5.1a}$$

$$E_{yy} = v_y + \frac{1}{2}(u_y^2 + v_y^2) \tag{3.5.1b}$$

$$2E_{xy} = u_y + v_x + (u_x u_y + v_x v_y) \tag{3.5.1c}$$

is referred to as *Green strain*. Green strain components E_{xx}, E_{yy}, $2E_{xy}$ are invariant with respect to rotation whereas Cauchy strain components ε_{xx}, ε_{yy}, ε_{2xy} are not (see Problem 1 and Problem 2). Thus, when a soft robot exhibits large rotational motion, we apply Green strain to formulate its deformation.

Green strain in three-dimensional deformation is given by $\boldsymbol{E} = [\, E_{xx},\, E_{yy},\, E_{zz},\, 2E_{yz},\, 2E_{zx},\, 2E_{xy}\,]^\top$, where

$$E_{xx} = u_x + \frac{1}{2}(u_x^2 + v_x^2 + w_x^2) \tag{3.5.2a}$$

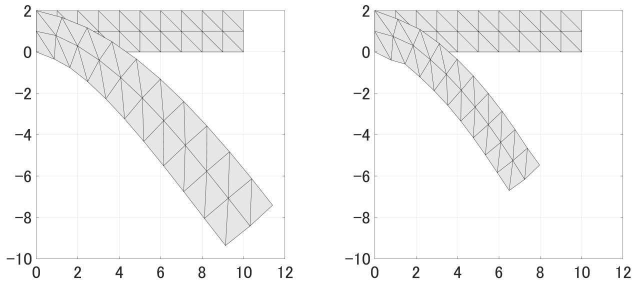
$$E_{yy} = v_y + \frac{1}{2}(u_y^2 + v_y^2 + w_y^2) \tag{3.5.2b}$$

$$E_{zz} = w_z + \frac{1}{2}(u_z^2 + v_z^2 + w_z^2) \tag{3.5.2c}$$

$$2E_{yz} = v_z + w_y + (u_y u_z + v_y v_z + w_y w_z) \tag{3.5.2d}$$

$$2E_{zx} = w_x + u_z + (u_z u_x + v_z v_x + w_z w_x) \tag{3.5.2e}$$

$$2E_{xy} = u_y + v_x + (u_x u_y + v_x v_y + w_x w_y) \tag{3.5.2f}$$



(a) Cauchy strain (b) Green strain

Figure 3.10: Computation via Cauchy and Green strains

Components E_{xx}, E_{yy}, E_{zz} correspond to normal deformation while $2E_{yz}$, $2E_{zx}$, $2E_{xy}$ represent shear deformation.

Assuming that the body material shows *linear isotropic elasticity* (see Section 7.1 for details), the strain energy density is formulated as follows:

$$\frac{1}{2}\boldsymbol{E}^\top(\lambda I_\lambda + \mu I_\mu)\boldsymbol{E} \tag{3.5.3}$$

Namely, simply replacing Cauchy strain $\boldsymbol{\varepsilon}$ in eq. (1.5.4) by Green strain $\boldsymbol{E}$ yields the strain energy density based on Green strain.

Comparing Cauchy and Green strains Let us compute the static deformation using Green strain $\boldsymbol{E} = [\,E_{xx},\, E_{yy},\, 2E_{xy}\,]^\top$ (eq. (3.5.1)) instead of Cauchy strain. Assuming that material exhibits linear isotropic elasticity, strain potential energy U_p stored in triangle $\mathrm{T}_p = \triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k$ is formulated as

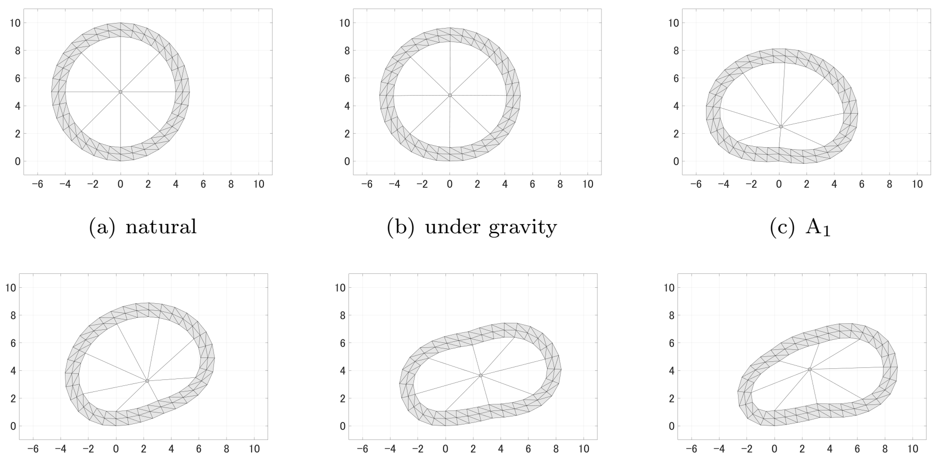
$$U_p = \frac{1}{2}\triangle h\left\{\lambda(E_{xx}+E_{yy})^2 + \mu(2E_{xx}^2 + 2E_{yy}^2 + (2E_{xy})^2)\right\} \tag{3.5.4}$$

where $\triangle = \triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k$. Procedure to calculate strain potential energy U_p is then summarized as follows:

function U_p = strain_potential_energy_in_triangle_based_on_Green_strain ($\boldsymbol{u}_\mathrm{N}$)
  Obtain displacements $\boldsymbol{u}_i$, $\boldsymbol{u}_j$, $\boldsymbol{u}_k$.
  Calculate vectors $\boldsymbol{a}$, $\boldsymbol{b}$ (eq. (2.3.11)) and partial derivatives u_x, u_y, v_x, v_y
  Calculate Green strain vector $\boldsymbol{E} = [\,E_{xx},\, E_{yy},\, 2E_{xy}\,]^\top$.
  Calculate potential energy U_p using eq. (3.5.4).

Summing up potential energies U_p yields total strain potential energy, implying that total strain potential energy U is a function of nodal displacement vector $\boldsymbol{u}_\mathrm{N}$. Consequently, solving conditional minimization problem eq. (3.1.1) directly using a conditional optimization algorithm, we obtain the static deformation based on Green strain.

Figure 3.10 demonstrates the difference between Cauchy and Green strains, by computing bending deformation of a beam of its length 10 cm, height 2 cm, and thickness 1 cm. One end of the beam is fixed to a wall while an external force of its magnitude 1.2 N is applied to the center of the other end. The external force acts parallel to the wall. Material of the beam exhibits isotropic linear elasticity, specified by $E = 0.1\,\mathrm{MPa}$ and $\nu = 0.48$. Figure 3.10(a) shows the computation based on Cauchy strain. The right end of the beam unnaturally



(a) natural (b) under gravity (c) A_1

(d) A_2 (e) A_2, A_6 (f) A_2, A_6, A_7

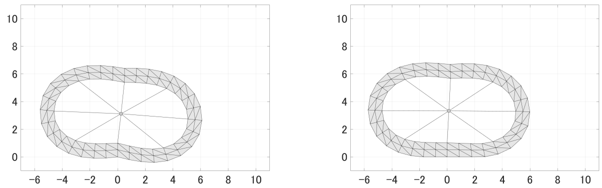
Figure 3.11: Elastic ring deformed by spring actuators

expands in the deformed shape. Note that triangle elements near the right end deform and rotate. Rotation affects Cauchy strain components, implying that rotation of the elements causes their unnatural deformation. Figure 3.10(b) shows the computation based on Green strain. We find that the beam bends naturally, avoiding the unnatural expansion of the elements. Recall that Green strain components are invariant against rotation. This rotation-invariance of Green strain results in accurate computation of deformation under finite rotation of elements.

Example (elastic ring with eight spring actuators) Let us calculate the deformation of an elastic ring driven by spring actuators. We focus on the two-dimensional deformation of cross-sectional area of the ring (Fig. 3.11(a)). The outer and inner radii of the ring are 5 mm and 4 mm in its natural shape. Material of the ring exhibits isotropic linear elasticity, specified by $E = 0.1\,\mathrm{MPa}$ and $\nu = 0.48$. Green strain is used during the calculation. Eight spring actuators labeled A_1 through A_8 counterclockwise are radially distributed inside the ring. A massless point supports the spring actuators, that is, one end of each spring actuator is connected to the massless point and the other end is connected to inner surface of the ring. In natural state, actuator A_1 is below the massless point. Let natural length of all actuators be $L = 4\,\mathrm{mm}$, that is, the natural length is equal to the inner radius of the ring. Let spring constant of all actuators be $k = 20\ \mathrm{N/m}$. The ring region is divided into 2×32 triangles. Nodal points P_1 and P_3 are bottom points of inner and outer surface of the ring region while P_2 is the midpoint of the two. Four constraints $u_3 = v_3 = 0$, $v_2 = 0$, and $v_1 = 0$ are imposed to avoid rigid body displacements.

Assume that gravitational force acts along the vertical direction downward. The elastic ring deforms under gravity (Fig. 3.11(b)). Spring actuators are able to generate shrinking forces. Let U_{strain} and U_{gravity} be strain potential energy and gravitational potential energy of an elastic ring, U_{springs} be the total potential energy of eight spring actuators, and W be work done by forces generated by the actuators. Internal energy of the system consisting of a elastic ring and spring actuators is then formulated as

$$I = U_{\mathrm{strain}} + U_{\mathrm{gravity}} + U_{\mathrm{springs}} - W$$



(a) A_1, A_5 (b) on floor

Figure 3.12: Deformed shape under inequality conditions

Letting $\boldsymbol{x}_{\text{point}}$ be positional vector of the massless point and P_j be nodal point on the inner surface of the elastic shell connected to the i-th spring actuator, and noting that positional vector of nodal point P_j is given by $\boldsymbol{x}_j + \boldsymbol{u}_j$, extension of the i-th actuator is given by

$$d_i = \|\boldsymbol{x}_{\text{point}} - (\boldsymbol{x}_j + \boldsymbol{u}_j)\| - L$$

Potential energy U_{springs} is then described as

$$U_{\text{springs}} = \sum_{i=1}^{8} \frac{1}{2} k d_i^2$$

Letting f_i be extensional force generated by the i-th actuator, work done by generated forces is described as

$$W = \sum_{i=1}^{8} f_i d_i$$

Four constraints $u_3 = v_3 = 0$, $v_2 = 0$, and $v_1 = 0$ are imposed to avoid rigid body displacements. Minimizing internal energy I under geometric constraints yields deformed shape of the elastic ring with eight spring actuators. Figure 3.11(c) represents the deformed shape of the elastic ring when actuator A_1 generates shrinking force of 5 N while Fig. 3.11(d) describes the deformation when actuator A_2 generate the same shrinking force. Activating multiple actuators yield complex deformation. Figure 3.11(e) shows the deformation when a pair of opposite actuators A_2 and A_6 generate shrinking force of 5 N. Figure 3.11(f) describes the deformation when three actuators A_2, A_6, and A_7 generate shrinking force of 5 N.

Inequality conditions Numerical optimization accepts not only equations but also inequalities as conditions. Figure 3.12(a) represents the deformed shape of the elastic ring when a pair of opposite actuators A_1 and A_5 generate shrinking force of 5 N. Several nodal points are below x-axis, that is, lie in region $y < 0$. Assume that the elastic ring deforms over a flat floor specified by x-axis. Then, all nodal points should lie in region $y \geq 0$. Noting that y-coordinate of nodal point P_k after deformation is given by $y_k + v_k$, we find one inequality condition $-v_k \leq y_k$. As this inequality is linear, combining all such inequality conditions yields a set of linear inequalities described as $A_{\text{ineq}}^{\top} \boldsymbol{u}_{\text{N}} \leq \boldsymbol{b}_{\text{ineq}}$, where A_{ineq} is a coefficient matrix and $\boldsymbol{b}_{\text{ineq}}$ is a constant vector.

Let $n = 3$ be the number of nodal points in the thickness direction and $m = 32$ the number of nodal points along the surfaces. Then, nodal points on the outer surface are P_n, P_{2n}, P_{3n} through P_{mn}. Here let us impose the following nine inequality conditions:

$$-v_k \leq y_k, \quad k = n,\, 2n,\, 3n,\, 4n,\, 5n,\, mn,\, (m-1)n,\, (m-2)n,\, (m-3)n$$

Point P_n is the bottom nodal point on the outer surface. Points P_{2n} through P_{5n} are on the right side of P_n while P_{mn} through $\mathrm{P}_{(m-3)n}$ are on the left side of P_n. The above inequality conditions imply that these nodal points should lie in region $y \geq 0$. Noting that dimension of $\boldsymbol{u}_{\mathrm{N}}$ is equal to $2mn$, coefficient matrix A_{ineq} is given as a $2mn \times 9$ matrix satisfying $A_{\mathrm{ineq}}^{\top} \boldsymbol{u}_{\mathrm{N}} = [\,-v_n,\, -v_{2n},\, -v_{3n},\, -v_{4n},\, -v_{5n},\, -v_{mn},\, -v_{(m-1)n},\, -v_{(m-2)n},\, -v_{(m-3)n}\,]^{\top}$. Constant vector is given as $\boldsymbol{b}_{\mathrm{ineq}} = [\,y_n,\, y_{2n},\, y_{3n},\, y_{4n},\, y_{5n},\, y_{mn},\, y_{(m-1)n},\, y_{(m-2)n},\, y_{(m-3)n}\,]^{\top}$. Figure 3.12(b) represents the deformed shape of the elastic ring under both equation and inequality conditions. We find that all nodal points lie in region $y \geq 0$.

3.6 Rectangular element

Let us approximate two-dimensional region S (Fig. 2.1(a)) by a set of small rectangles. Assume that edges of rectangles are parallel to x- or y-axes. Dimensions of all rectangles are identical; l_x be the length of edges parallel to x-axis and l_y be the length of edges parallel to y-axis. First, let us calculate potential energy stored in a rectangular element $\mathrm{R}_p = \square \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k\mathrm{P}_l$. Let us introduce the following four functions:

$$N_i^p = \frac{(x_i + l_x - x)(y_i + l_y - y)}{l_x l_y}, \qquad N_j^p = \frac{(x - x_i)(y_i + l_y - y)}{l_x l_y},$$
$$N_k^p = \frac{(x - x_i)(y - y_i)}{l_x l_y}, \qquad N_l^p = \frac{(x_i + l_x - x)(y - y_i)}{l_x l_y}.$$

Note that $N_i^p = 1$ at nodal point P_i while $N_i^p = 0$ at the other nodal points P_j, P_k, P_l. Let $\boldsymbol{u}_i$, $\boldsymbol{u}_j$, $\boldsymbol{u}_k$, $\boldsymbol{u}_l$ be displacement vectors at four nodal points P_i, P_j, P_k, P_l. Piecewise bilinear approximation of function $\boldsymbol{u}$ over the rectangular region is then described as

$$\boldsymbol{u} = \boldsymbol{u}_i N_i^p + \boldsymbol{u}_j N_j^p + \boldsymbol{u}_k N_k^p + \boldsymbol{u}_l N_l^p \tag{3.6.1}$$

Introducing collective vectors $\boldsymbol{\gamma}_u = [\,u_i,\, u_j,\, u_k,\, u_l\,]^{\top}$ and $\boldsymbol{\gamma}_v = [\,v_i,\, v_j,\, v_k,\, v_l\,]^{\top}$, we find

$$\frac{\partial u}{\partial x} = \boldsymbol{a}^{\top}\boldsymbol{\gamma}_u, \qquad \frac{\partial u}{\partial y} = \boldsymbol{b}^{\top}\boldsymbol{\gamma}_u, \qquad \frac{\partial v}{\partial x} = \boldsymbol{a}^{\top}\boldsymbol{\gamma}_v, \qquad \frac{\partial v}{\partial y} = \boldsymbol{b}^{\top}\boldsymbol{\gamma}_v$$

where

$$\boldsymbol{a} = \frac{1}{l_x l_y}\begin{bmatrix} -(y_i + l_y - y) \\ y_i + l_y - y \\ y - y_i \\ -(y - y_i) \end{bmatrix}, \qquad \boldsymbol{b} = \frac{1}{l_x l_y}\begin{bmatrix} -(x_i + l_x - x) \\ -(x - x_i) \\ x - x_i \\ x_i + l_x - x \end{bmatrix}$$

Note that vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ depend on x and y.

Letting $\boldsymbol{\gamma} = [\,\boldsymbol{\gamma}_u^{\top},\, \boldsymbol{\gamma}_v^{\top}\,]^{\top}$ and applying calculation process described in Section 2.3, we obtain potential energy stored in rectangular element $\mathrm{R}_p = \square \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k\mathrm{P}_l$:

$$U_p = \frac{1}{2}\boldsymbol{\gamma}^{\top}\left(\lambda H_{\lambda} + \mu H_{\mu}\right)\boldsymbol{\gamma} \tag{3.6.2}$$

where

$$H_{\lambda} = \int_{x_i}^{x_i + l_x}\int_{y_i}^{y_i + l_y}\begin{bmatrix} \boldsymbol{a}\boldsymbol{a}^{\top} & \boldsymbol{a}\boldsymbol{b}^{\top} \\ \boldsymbol{b}\boldsymbol{a}^{\top} & \boldsymbol{b}\boldsymbol{b}^{\top} \end{bmatrix} h \,\mathrm{dS},$$
$$H_{\mu} = \int_{x_i}^{x_i + l_x}\int_{y_i}^{y_i + l_y}\begin{bmatrix} 2\boldsymbol{a}\boldsymbol{a}^{\top} + \boldsymbol{b}\boldsymbol{b}^{\top} & \boldsymbol{b}\boldsymbol{a}^{\top} \\ \boldsymbol{a}\boldsymbol{b}^{\top} & 2\boldsymbol{b}\boldsymbol{b}^{\top} + \boldsymbol{a}\boldsymbol{a}^{\top} \end{bmatrix} h \,\mathrm{dS}.$$

Converting variables x and y into $\xi = x - x_i$ and $\eta = y - y_i$, we have

$$H_\lambda = h \int_0^{l_x} \mathrm{d}\xi \int_0^{l_y} \mathrm{d}\eta \begin{bmatrix} \boldsymbol{\alpha}\boldsymbol{\alpha}^\top & \boldsymbol{\alpha}\boldsymbol{\beta}^\top \\ \boldsymbol{\beta}\boldsymbol{\alpha}^\top & \boldsymbol{\beta}\boldsymbol{\beta}^\top \end{bmatrix},$$

$$H_\mu = h \int_0^{l_x} \mathrm{d}\xi \int_0^{l_y} \mathrm{d}\eta \begin{bmatrix} 2\boldsymbol{\alpha}\boldsymbol{\alpha}^\top + \boldsymbol{\beta}\boldsymbol{\beta}^\top & \boldsymbol{\beta}\boldsymbol{\alpha}^\top \\ \boldsymbol{\alpha}\boldsymbol{\beta}^\top & 2\boldsymbol{\beta}\boldsymbol{\beta}^\top + \boldsymbol{\alpha}\boldsymbol{\alpha}^\top \end{bmatrix}$$

where

$$\boldsymbol{\alpha} = \frac{1}{l_x l_y} \begin{bmatrix} -(l_y - \eta) \\ l_y - \eta \\ \eta \\ -\eta \end{bmatrix}, \qquad \boldsymbol{\beta} = \frac{1}{l_x l_y} \begin{bmatrix} -(l_x - \xi) \\ -\xi \\ \xi \\ l_x - \xi \end{bmatrix}.$$

Calculating the above integrals, we find

$$H_\lambda = \begin{bmatrix} H_\lambda^{uu} & H_\lambda^{uv} \\ H_\lambda^{vu} & H_\lambda^{vv} \end{bmatrix}, \qquad H_\mu = \begin{bmatrix} 2H_\lambda^{uu} + H_\lambda^{vv} & H_\lambda^{vu} \\ H_\lambda^{uv} & 2H_\lambda^{vv} + H_\lambda^{uu} \end{bmatrix}, \tag{3.6.3}$$

where

$$H_\lambda^{uu} = \frac{h\, l_y}{6\, l_x} \begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix}, \qquad H_\lambda^{uv} = \frac{h}{4} \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$H_\lambda^{vu} = (H_\lambda^{uv})^\top, \qquad H_\lambda^{vv} = \frac{h\, l_x}{6\, l_y} \begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix}$$

(see Problem 3). Matrices H_λ and H_μ depend on l_x and l_y but are independent of $\boldsymbol{x}_i$.

Let us permute rows and columns of H_λ and H_μ so that U_p is described by a quadratic form with respect to $\boldsymbol{u}_p = [\, \boldsymbol{u}_i^\top, \boldsymbol{u}_j^\top, \boldsymbol{u}_k^\top, \boldsymbol{u}_l^\top \,]^\top$. That is

$$\boldsymbol{\gamma}^\top H_\lambda \boldsymbol{\gamma} = \boldsymbol{u}_p^\top\, J_\lambda^p\, \boldsymbol{u}_p, \qquad \boldsymbol{\gamma}^\top H_\mu \boldsymbol{\gamma} = \boldsymbol{u}_p^\top\, J_\mu^p\, \boldsymbol{u}_p$$

Permuting rows and columns so that $1, 5, 2, 6, 3, 7, 4, 8$ rows and columns of H_λ be 1 through 8 rows and columns of J_λ^p, we obtain matrix J_λ^p. Similarly, permuting rows and columns so that $1, 5, 2, 6, 3, 7, 4, 8$ rows and columns of H_μ be 1 through 8 rows and columns of J_μ^p, we obtain matrix J_μ^p. Finally, we find strain potential energy stored in $\mathrm{R}_p = \Box \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l$:

$$U_p = \frac{1}{2} \boldsymbol{u}_p^\top\, K_p\, \boldsymbol{u}_p \tag{3.6.4}$$

where

$$K_p = \lambda J_\lambda^p + \mu J_\mu^p \tag{3.6.5}$$

is referred to as *partial stiffness matrix*. Synthesizing partial stiffness matrices of individual rectangles yields the stiffness matrix of the body:

$$K = \bigoplus_p K_p. \tag{3.6.6}$$

Note that operator $\oplus$ works block-wise.

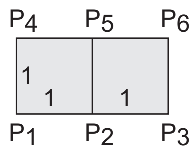


Figure 3.13: Region consisting of rectangular elements

Example Let us calculate the stiffness matrix of an elastic body consisting of two rectangles $\mathrm{R}_1 = \square \mathrm{P}_1\mathrm{P}_2\mathrm{P}_5\mathrm{P}_4$ and $\mathrm{R}_2 = \square \mathrm{P}_2\mathrm{P}_3\mathrm{P}_6\mathrm{P}_5$ (Fig. 3.13). Stiffness matrix of the body is then given by

$$K = K_1 \oplus K_2 = K_{1,2,5,4} \oplus K_{2,3,6,5},$$

where K consists of 6^2 2×2 blocks. In this example, $(1,2,3,4) \times (1,2,3,4)$ blocks of $K_1 = K_{1,2,5,4}$ contribute to $(1,2,5,4) \times (1,2,5,4)$ blocks of K and $(1,2,3,4) \times (1,2,3,4)$ blocks of $K_2 = K_{2,3,6,5}$ contribute to $(2,3,6,5) \times (2,3,6,5)$ blocks of K. Connection matrices are given by

$$J_\lambda = \frac{1}{12}\left[\begin{array}{cc|cc|cc|cc|cc|cc}
4 & 3 & -4 & 3 & & & 2 & -3 & -2 & -3 & & \\
3 & 4 & -3 & 2 & & & 3 & -4 & -3 & -2 & & \\ \hline
-4 & -3 & 8 & 0 & -4 & 3 & -2 & 3 & 4 & 0 & -2 & -3 \\
3 & 2 & 0 & 8 & -3 & 2 & 3 & -2 & 0 & -8 & -3 & -2 \\ \hline
 & & -4 & -3 & 4 & -3 & & & -2 & 3 & 2 & 3 \\
 & & 3 & 2 & -3 & 4 & & & 3 & -2 & -3 & -4 \\ \hline
2 & 3 & -2 & 3 & & & 4 & -3 & -4 & -3 & & \\
-3 & -4 & 3 & -2 & & & -3 & 4 & 3 & 2 & & \\ \hline
-2 & -3 & 4 & 0 & -2 & 3 & -4 & 3 & 8 & 0 & -4 & -3 \\
-3 & -2 & 0 & -8 & 3 & -2 & -3 & 2 & 0 & 8 & 3 & 2 \\ \hline
 & & -2 & -3 & 2 & -3 & & & -4 & 3 & 4 & 3 \\
 & & -3 & -2 & 3 & -4 & & & -3 & 2 & 3 & 4
\end{array}\right]$$

$$J_\mu = \frac{1}{4}\left[\begin{array}{cc|cc|cc|cc|cc|cc}
4 & 1 & -2 & -1 & & & 0 & 1 & -2 & -1 & & \\
1 & 4 & 1 & 0 & & & -1 & -2 & -1 & -2 & & \\ \hline
-2 & 1 & 8 & 0 & -2 & -1 & -2 & 1 & 0 & 0 & -2 & -1 \\
-1 & 0 & 0 & 8 & 1 & 0 & 1 & -2 & 0 & -4 & -1 & -2 \\ \hline
 & & -2 & 1 & 4 & -1 & & & -2 & 1 & 0 & -1 \\
 & & -1 & 0 & -1 & 4 & & & 1 & -2 & 1 & -2 \\ \hline
0 & -1 & -2 & 1 & & & 4 & -1 & -2 & 1 & & \\
1 & -2 & 1 & -2 & & & -1 & 4 & -1 & 0 & & \\ \hline
-2 & -1 & 0 & 0 & -2 & 1 & -2 & -1 & 8 & 0 & -2 & 1 \\
-1 & -2 & 0 & -4 & 1 & -2 & 1 & 0 & 0 & 8 & -1 & 0 \\ \hline
 & & -2 & -1 & 0 & 1 & & & -2 & -1 & 4 & 1 \\
 & & -1 & -2 & -1 & -2 & & & 1 & 0 & 1 & 4
\end{array}\right]$$

When Lamé's constants are uniform over the region, stiffness matrix is given by $K = \lambda J_\lambda + \mu J_\mu$.

Implementation Class `Rectangle` was introduced to define rectangular elements. Class `Rectangle` involves the following methods:

partial_stiffness_matrix calculating partial stiffness matrix of the rectangle

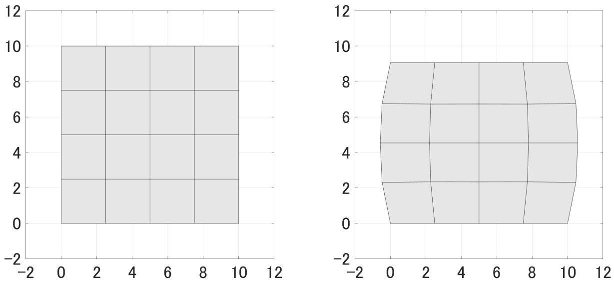


Figure 3.14: Deformation in push test under rectangle elements

partial_inertia_matrix calculating partial inertia matrix of the rectangle
partial_strain_potential_energy strain potential energy stored in the rectangle
partial_strain_potential_energy_Green_strain strain energy using Green strain

The following methods are implemented in class `Body`:

rectangle_elements defines rectangular elements in the body

Let us define an elastic body given in Fig. 3.13. Coordinates of individual nodal points are listed as

$$\text{points} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 & \boldsymbol{x}_4 & \boldsymbol{x}_5 & \boldsymbol{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Nodal point numbers for individual rectangular elements are listed as

$$\text{rectangles} = \begin{bmatrix} 1 & 2 & 5 & 4 \\ 2 & 3 & 6 & 5 \end{bmatrix}.$$

The body region is then given by

```
elastic = Body(6, points, [], [], thickness);
elastic = elastic.rectangle_elements(2, rectangles);
```

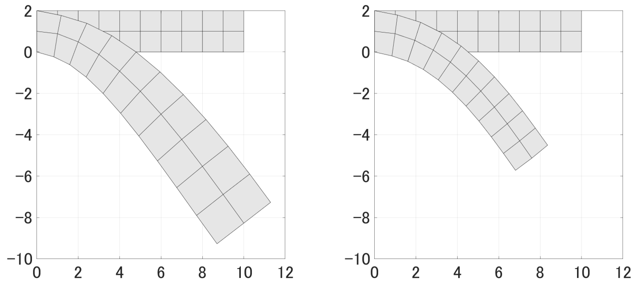
where `thickness` specifies thickness h of the two-dimensional body.

Deformation in push test (Fig. 3.4) was calculated based on rectangular elements. Figure 3.14 shows the computation result. It turns out that the results are almost similar to each other; Fig. 3.14 shows more symmetric deformed shape.

Beam deformation under an external force (Fig. 3.10) was calculated based on rectangular elements. Strain potential energy U_p stored in rectangle R_p using Green strain is described as

$$U_p = h \int_0^{l_x} \mathrm{d}\xi \int_0^{l_y} \mathrm{d}\eta \; \frac{1}{2} \left\{ \lambda (E_{xx} + E_{yy})^2 + \mu (2E_{xx}^2 + 2E_{yy}^2 + (2E_{xy})^2) \right\},$$

of which value can be computed by a numerical integration algorithm. Since total strain potential energy is given by summing up potential energies U_p, static deformation based on Green strain can be calculated by solving conditional minimization problem eq. (3.1.1) directly using a conditional optimization algorithm. Figure 3.15(a) shows the computation based on Cauchy strain while Fig. 3.15(b) shows the computation based on Green strain. Calculation based on Green strain exhibits more accurate deformation.



(a) Cauchy strain (b) Green strain

Figure 3.15: Calculation via Cauchy and Green strains using rectangular elements

Example (deformation by inner pressure) Inner pressure inside an elastic body causes its deformation. Deformation of layered elastic body (Fig. 3.16(a)) was calculated. The body of its width 10 cm, height 2 cm, thickness $h = 1\,\mathrm{cm}$, Young's modulus $E = 0.1\,\mathrm{MPa}$, and Poisson's ratio $\nu = 0.48$ consists of two layers. Bottom layer expands due to its inner pressure p. The body is modeled by 10×2 rectangular elements. Green strain is applied to calculation of strain potential energy.

Area of deformed shape of rectangle $\mathrm{R}_p = \square \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l$ is described as

$$S_p(\boldsymbol{u}_\mathrm{N}) = \frac{1}{2} \left| \begin{matrix} \boldsymbol{x}_{i,k} + \boldsymbol{u}_k - \boldsymbol{u}_i & \boldsymbol{x}_{j,l} + \boldsymbol{u}_l - \boldsymbol{u}_j \end{matrix} \right| \tag{3.6.7}$$

where $\boldsymbol{x}_{i,k} = \boldsymbol{x}_k - \boldsymbol{x}_i = [\, l_x,\, l_y \,]^\top$ and $\boldsymbol{x}_{j,l} = \boldsymbol{x}_l - \boldsymbol{x}_j = [\, -l_x,\, l_y \,]^\top$ (see Problem 4). Increase of the area is then given by $\Delta S_p(\boldsymbol{u}_\mathrm{N}) = S_p(\boldsymbol{u}_\mathrm{N}) - l_x l_y$, resulting the increase of total area as

$$\Delta S(\boldsymbol{u}_\mathrm{N}) = \sum_p \Delta S_p(\boldsymbol{u}_\mathrm{N}).$$

Thus, we have the following minimization problem:

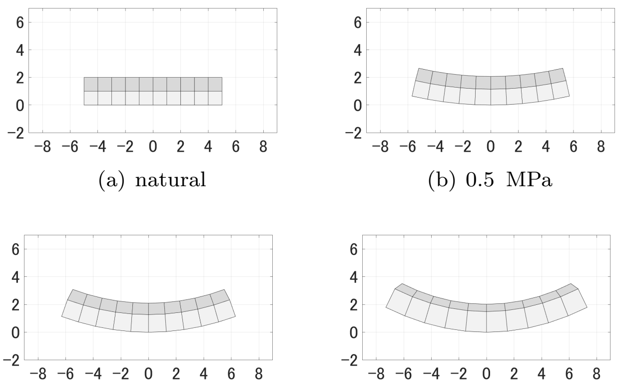
$$\begin{aligned} &\text{minimize} \quad I(\boldsymbol{u}_\mathrm{N}) = \frac{1}{2} \boldsymbol{u}_\mathrm{N}^\top K \, \boldsymbol{u}_\mathrm{N} - ph \, \Delta S(\boldsymbol{u}_\mathrm{N}) \\ &\text{subject to} \quad A^\top \boldsymbol{u}_\mathrm{N} = \boldsymbol{0} \end{aligned} \tag{3.6.8}$$

Three constraints $u_6 = v_6 = 0$ and $v_{17} = 0$ are imposed to avoid rigid body displacements. Figs. 3.16(b), 3.16(c), and 3.16(d) show computation results at inner pressure $p = 0.5\,\mathrm{MPa}$, $1.0\,\mathrm{MPa}$, and $2.0\,\mathrm{MPa}$. As shown in the figures, the body is curved due to expansion of the bottom layer.

Inertia matrix Applying calculation process described in Section 2.3, we obtain partial inertia matrix of rectangular element $\mathrm{R}_p = \square \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k \mathrm{P}_l$:

$$M_p = \frac{\rho h l_x l_y}{36} \begin{bmatrix} 4I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} \\ 2I_{2\times 2} & 4I_{2\times 2} & 2I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & 2I_{2\times 2} & 4I_{2\times 2} & 2I_{2\times 2} \\ 2I_{2\times 2} & I_{2\times 2} & 2I_{2\times 2} & 4I_{2\times 2} \end{bmatrix} \tag{3.6.9}$$

(see Problem 5). Note that the sum of all blocks of matrix M_p is equal to $\rho h l_x l_y \, I_{2\times 2}$, which denotes the mass of a rectangular element. Synthesizing partial inertia matrices of individual



(a) natural

(b) 0.5 MPa

(c) 1.0 MPa

(d) 2.0 MPa

Figure 3.16: Deformation by inner pressure of layered body

rectangles yields the inertia matrix of the body:

$$M = \bigoplus_{p} M_p. \tag{3.6.10}$$

Note that operator $\oplus$ works block-wise.

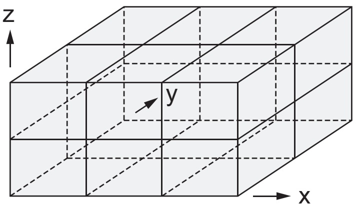
3.7 Static deformation of three-dimensional soft body

Numbering in nodal points and tetrahedra Let us number nodal points and tetrahedra of a cuboidal body. Figure 3.17(a) shows a cuboidal body, consisting of $4 \times 3 \times 3$ nodal points and $6 \times (3 \times 2 \times 2)$ tetrahedra. Consecutive numbers are assigned to nodal points. Bottom face includes P_1 through P_{12} (Fig. 3.17(b)) and top face includes P_{25} through P_{36} (Fig. 3.17(b)). The cuboidal body consists of $3 \times 2 \times 2$ divided cuboids. Each divided cuboid consists of six tetrahedra. Bottom-front-left divided cuboid has six tetrahedra T_1 through T_6 and bottom-front-right divided cuboid has six tetrahedra T_{13} through T_{18}. Top-back-right divided cuboid has six tetrahedra T_{67} through T_{72}.

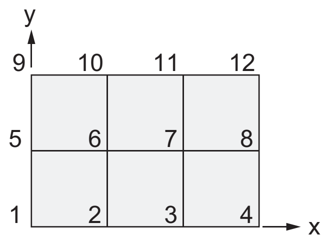
Let P_i, P_j, P_k, P_l, P_m, P_n, P_r, and P_s be vertices of a cuboid (Fig. 3.18(a)). This cuboid can be divided into six tetrahedra: $\Diamond P_j P_m P_l P_i$, $\Diamond P_m P_j P_l P_s$, $\Diamond P_m P_j P_s P_n$, $\Diamond P_s P_k P_j P_l$, $\Diamond P_s P_k P_n P_j$, and $\Diamond P_n P_k P_s P_r$ (Fig. 3.18(b)).

Twisting a beam Let us calculate the twisting of a elastic beam. Figure 3.19(a) shows an elastic beam of length 4 cm with cross-section of square of its side 1 cm. The beam is divided into 6×4 tetrahedra with $2 \times 2 \times 5$ nodal points. Elasticity is specified by Young's modulus $E = 0.1$ MPa and Poisson's ratio $\nu = 0.48$. The bottom face is fixed to the ground and the top face is rotated around its center by 20 degrees. Figure 3.19(b) shows the calculated result based on Cauchy strain and Fig. 3.19(c) is based on Green strain. Both results appropriately describe the twisted deformation of an elastic beam.

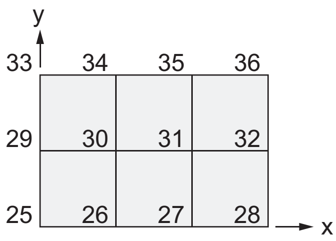
Expansion of a membrane Let us calculate nodal forces equivalent to pressure application. Assume that pressure p is applied to triangle $\triangle P_i P_j P_k$. Magnitude of the equivalent force is given by $p\triangle$, where $\triangle$ denotes the area of triangle $\triangle P_i P_j P_k$. Since the equivalent



(a) division of cuboidal body

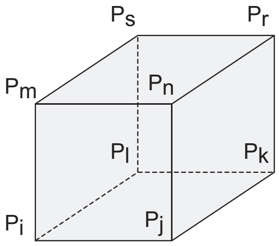


(b) bottom face

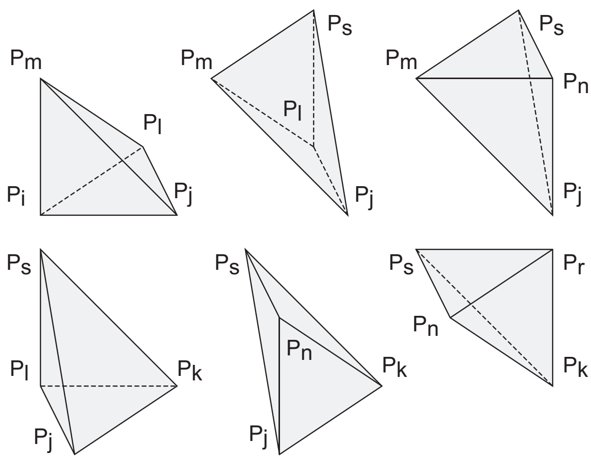


(c) top face

Figure 3.17: Cuboidal body



(a) cuboid



(b) six tetrahedra

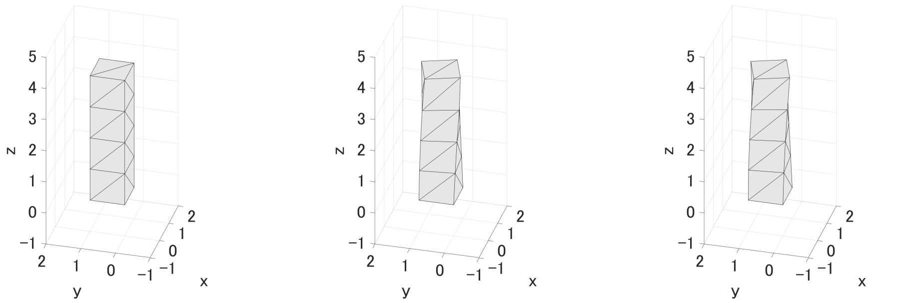
Figure 3.18: Division of a cuboid into six tetrahedra

force is perpendicular to the triangle, normal directional vector of the equivalent force is given by

$$\boldsymbol{d} = \frac{(\boldsymbol{r}_j - \boldsymbol{r}_i) \times (\boldsymbol{r}_k - \boldsymbol{r}_i)}{2\triangle},$$

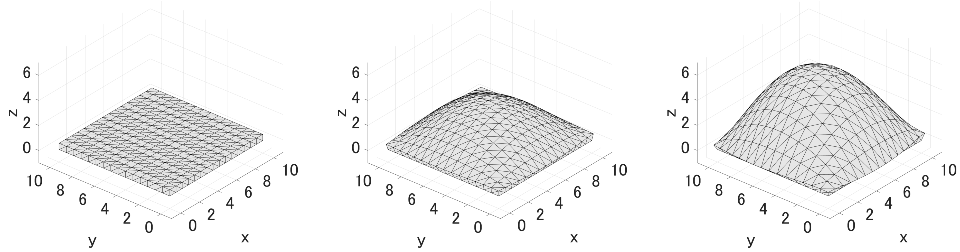
where $\boldsymbol{r}_i = \boldsymbol{x}_i + \boldsymbol{u}_i$, $\boldsymbol{r}_j = \boldsymbol{x}_j + \boldsymbol{u}_j$, and $\boldsymbol{r}_k = \boldsymbol{x}_k + \boldsymbol{u}_k$ represent current position of P_i, P_j, and P_k. The equivalent force $p\triangle\boldsymbol{d}$ is equally distributed to the three nodal points. Thus, nodal forces at P_i, P_j, and P_k are described as:

$$\boldsymbol{f}_i = \boldsymbol{f}_j = \boldsymbol{f}_k = \frac{1}{6} p (\boldsymbol{r}_j - \boldsymbol{r}_i) \times (\boldsymbol{r}_k - \boldsymbol{r}_i) \tag{3.7.1}$$



(a) natural (b) deformed (Cauchy strain) (c) deformed (Green strain)

Figure 3.19: Twisted beam



(a) natural (b) 40 kPa (c) 80 kPa

Figure 3.20: Expansion of an elastic membrane under pressure

(see Problem 9). Let S be a set of triangles where pressure p is applied. Nodal force vector equivalent to pressure application is then described as follows:

$$\boldsymbol{f}_{\rm N} = \bigoplus_{\triangle {\rm P}_i {\rm P}_j {\rm P}_k \in S} \begin{bmatrix} \boldsymbol{f}_i \\ \boldsymbol{f}_j \\ \boldsymbol{f}_k \end{bmatrix}. \tag{3.7.2}$$

Let us apply the following conditional optimization problem to compute the expansion of a membrane:

$$\begin{aligned} &\text{minimize} \quad I = \frac{1}{2}\, \boldsymbol{u}_{\rm N}^{\top}\, K\, \boldsymbol{u}_{\rm N} - \boldsymbol{f}_{\rm N} \\ &\text{subject to} \quad \begin{bmatrix} K & -A \\ -A^{\top} & \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{\rm N} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_{\rm N} \\ \boldsymbol{0} \end{bmatrix} \end{aligned} \tag{3.7.3}$$

where $\boldsymbol{f}_{\rm N}$ is a collective vector of nodal forces equivalent to pressure application. Note that $\boldsymbol{f}_{\rm N}$ is not constant but quadratic with respect to elements of $\boldsymbol{u}_{\rm N}$. The above problem implies that internal energy I should be minimized under equilibrium equation $-K\boldsymbol{u}_{\rm N} + A\boldsymbol{\lambda} + \boldsymbol{f}_{\rm N} = \boldsymbol{0}$ and a set of geometric constraints $A^{\top}\boldsymbol{u}_{\rm N} = \boldsymbol{0}$.

Let us calculate the expansion of a square membrane of side 10 cm and thickness 5 mm (Fig. 3.20(a)). The membrane is divided into $6 \times (20 \times 20 \times 1)$ tetrahedra with $21 \times 21 \times 2$ nodal points. Elasticity is specified by Young's modulus $E = 0.1$ MPa and Poisson's ratio $\nu = 0.48$. Cauchy strain is applied to the calculation. The boundary of the bottom face is fixed to the ground and pressure p is applied to the bottom surface. Calculated shapes of the expanding membrane at $p = 40\,\mathrm{kPa}$ and $p = 80\,\mathrm{kPa}$ are shown in Figs. 3.20(b) and 3.20(c). Expansion under pressure can be calculated appropriately. This computation was performed by MATLAB running on Windows 10, i5–6300U CPU at 2.40 GHz with 8.0 GB memory. Computation time was about 9.5 minutes.

Problems

1. Show that Green strain components are invariant with respect to rotation whereas Cauchy strain components are not.

2. Let $\mathrm{P}(x, y)$ and $\mathrm{Q}(x + \delta x, y + \delta y)$ are neighboring points. Let δs be the distance between P and Q in the natural shape and $\delta s'$ be the distance in the deformed shape. Show that

$$(\delta s')^2 - (\delta s)^2 = 2 \begin{bmatrix} \delta x & \delta y \end{bmatrix} \begin{bmatrix} E_{xx} & E_{xy} \\ E_{xy} & E_{yy} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

   where E_{xx}, E_{yy}, E_{xy} are Green strain components.

3. Show eq. (3.6.3).

4. Show eq. (3.6.7).

5. Show eq. (3.6.9).

6. Show that area of polygon $\mathrm{P}_1\mathrm{P}_2 \cdots \mathrm{P}_n$ is given by

$$S = \frac{1}{2} \sum_{k=1}^{n} (x_k - x_{k+1})(y_k + y_{k+1}),$$

   where (x_k, y_k) denotes the coordinates of point P_k and $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. Additionally

$$\frac{\partial S}{\partial x_k} = \frac{1}{2}(-y_{k-1} + y_{k+1}), \qquad \frac{\partial S}{\partial y_k} = \frac{1}{2}(x_{k-1} - x_{k+1})$$

7. Compute the static deformation of a beam under gravity. Apply geometric and physical parameters used in computation of Fig. 3.10. Apply Cauchy strain and Green strain to compare the obtained results.

8. Let $\boldsymbol{u}_i = [\, u_i,\, v_i \,]^\top$, $\boldsymbol{u}_j = [\, u_j,\, v_j \,]^\top$, and $\boldsymbol{u}_k = [\, u_k,\, v_k \,]^\top$ be displacement vectors of P_i, P_j, and P_k. Let U_p be Green strain based potential energy stored in triangle $\mathrm{T}_p = \triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k$ (eq. (3.5.4)). Introduce collective vectors $\boldsymbol{\gamma}_u = [\, u_i,\, u_j,\, u_k \,]^\top$ and $\boldsymbol{\gamma}_v = [\, v_i,\, v_j,\, v_k \,]^\top$. Calculate partial derivatices $\partial U_p / \partial \boldsymbol{\gamma}_u$ and $\partial U_p / \partial \boldsymbol{\gamma}_v$. These partial derivatices directly yield $\partial U_p / \partial \boldsymbol{u}_i$, $\partial U_p / \partial \boldsymbol{u}_j$, and $\partial U_p / \partial \boldsymbol{u}_k$.

9. Let V be the volume of a polyhedron. Let P_i be a vertex of the polyhedra and $\boldsymbol{x}_i$ be the positional vector of P_i. Calculate partial derivative $\partial V / \partial \boldsymbol{x}_i$.