Chapter 4

Computing Dynamic Deformation

4.1 Variational principle in dynamics

Let us calculate the dynamic deformation of an elastic body. We apply variational principle in dynamics for the calculation. Let T and U be kinetic and potential energies of the body. External forces applied to the body will deform the body. Let W be work done by external forces. Geometric constraints imposed on the body causes the deformation of the body. Let $\boldsymbol{R}$ be a collective vector of geometric constraints. Variational principle in dynamics insists that a geometrically admissible motion of a holonomic system between two configurations at specified times is natural if and only if the variation of action integral vanishes for any variations. This is equivalent to the Lagrange equations of motions. Lagrangian of a system is defined as

$$\mathcal{L} = T - U + W + \boldsymbol{\lambda}^{\top}\boldsymbol{R} \tag{4.1.1}$$

where $\boldsymbol{\lambda}$ denote a collective vector consisting of Lagrange multipliers corresponding to individual constraints.

In finite element approximation, deformation of an elastic body is described by nodal displacement vector $\boldsymbol{u}_{\mathrm{N}}$ and its time-derivative $\dot{\boldsymbol{u}}_{\mathrm{N}}$, implying that the Lagrangian is a function of vector $\boldsymbol{u}_{\mathrm{N}}$ and $\dot{\boldsymbol{u}}_{\mathrm{N}}$. Lagrange equation of motion and deformation is then described as follows:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}_{\mathrm{N}}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}_{\mathrm{N}}} = \mathbf{0} \tag{4.1.2}$$

A set of constraints $\boldsymbol{R} = \mathbf{0}$ can be converted into a set of ordinary differential equations stabilizing the constraints:

$$\ddot{\boldsymbol{R}} + 2\alpha\dot{\boldsymbol{R}} + \alpha^2\boldsymbol{R} = \mathbf{0} \tag{4.1.3}$$

where α is a positive constant. By solving the above two ordinary differential equations, we can compute dynamic deformation of a body.

Many algorithms for solving a set of ordinary differential equations (ODEs) have been proposed and available. We can apply such ODE solvers to the above ordinary differential equations. For example, MATLAB offers ODE solvers such as `ode45` and `ode15s`. Solving a set of ordinary differential equations, we can obtain $\boldsymbol{u}_{\mathrm{N}}(t)$, which sketches dynamic deformation of a body during a given time period.

4.2 Dynamic deformation of one-dimensional soft body

Let us formulate the dynamic deformation of an regular-shaped elastic beam of its length L. Assume that cross-sectional area A, Young's modulus E, and density ρ are uniform along the beam, implying that they are constants . Dividing $[\,0,\,L\,]$ into four small regions, kinetic and strain potential energies of the beam are described as follows:

$$T = \frac{1}{2}\,\dot{\boldsymbol{u}}_{\mathrm{N}}^{\top}\,M\,\dot{\boldsymbol{u}}_{\mathrm{N}}, \qquad U = \frac{1}{2}\,\boldsymbol{u}_{\mathrm{N}}^{\top}\,K\,\boldsymbol{u}_{\mathrm{N}}$$

(see eqs. (2.2.16)(2.2.6)), where inertia matrix M is given in eq. (2.2.17) and stiffness matrix K is described in eq. (2.2.5). Assume that end point $\mathrm{P}(0)$ is fixed to space while an external force f is applied to end point $\mathrm{P}(L)$. Work done by the external force is then described as

$$W = \boldsymbol{f}^{\top}\boldsymbol{u}_{\mathrm{N}}$$

where $\boldsymbol{f} = [\,0,\,0,\,0,\,0,\,f\,]^{\top}$. Since displacement of point $\mathrm{P}(0)$ should be equal to zero, the following geometric constraint must be satisfied:

$$R = \boldsymbol{a}^{\top}\boldsymbol{u}_{\mathrm{N}} = 0$$

where $\boldsymbol{a} = [\,1,\,0,\,0,\,0,\,0\,]^{\top}$. Consequently, we have the following Lagrangian:

$$\mathcal{L}(\boldsymbol{u},\dot{\boldsymbol{u}}) = \frac{1}{2}\dot{\boldsymbol{u}}_{\mathrm{N}}^{\top}\,M\,\dot{\boldsymbol{u}}_{\mathrm{N}} - \frac{1}{2}\,\boldsymbol{u}_{\mathrm{N}}^{\top}\,K\,\boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f}^{\top}\boldsymbol{u}_{\mathrm{N}} + \lambda_a\boldsymbol{a}^{\top}\boldsymbol{u}_{\mathrm{N}} = \boldsymbol{0} \tag{4.2.1}$$

where λ_a is a Lagrange multiplier corresponding to a single constraint $\boldsymbol{a}^{\top}\boldsymbol{u}_{\mathrm{N}} = 0$. Since M and K are constant matrices, we have

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{u}_{\mathrm{N}}} = -K\boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f} + \lambda_a\boldsymbol{a}, \qquad \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{u}}_{\mathrm{N}}} = M\dot{\boldsymbol{u}}_{\mathrm{N}}$$

which directly yields

$$-K\boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f} + \lambda_a\boldsymbol{a} - M\ddot{\boldsymbol{u}}_{\mathrm{N}} = \boldsymbol{0} \tag{4.2.2}$$

Equation for stabilizing constraint $\boldsymbol{a}^{\top}\boldsymbol{u}_{\mathrm{N}} = 0$ is given by

$$\boldsymbol{a}^{\top}\ddot{\boldsymbol{u}}_{\mathrm{N}} + 2\alpha\boldsymbol{a}^{\top}\dot{\boldsymbol{u}}_{\mathrm{N}} + \alpha^2\boldsymbol{a}^{\top}\boldsymbol{u}_{\mathrm{N}} = \boldsymbol{0} \tag{4.2.3}$$

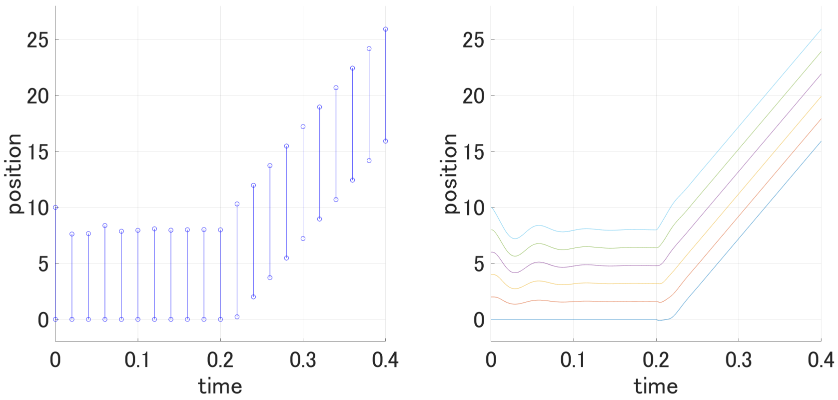
where α is a positive constant. Introducing $\boldsymbol{v}_{\mathrm{N}} = \dot{\boldsymbol{u}}_{\mathrm{N}}$, the above two ordinary differential equations turn into

$$\begin{aligned} M\dot{\boldsymbol{v}}_{\mathrm{N}} - \boldsymbol{a}\lambda_a &= -K\boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f} \\ -\boldsymbol{a}^{\top}\dot{\boldsymbol{v}}_{\mathrm{N}} &= 2\alpha\boldsymbol{a}^{\top}\boldsymbol{v}_{\mathrm{N}} + \alpha^2\boldsymbol{a}^{\top}\boldsymbol{u}_{\mathrm{N}} \end{aligned}$$

Combining the above two equations, we have

$$\begin{bmatrix} M & -\boldsymbol{a} \\ -\boldsymbol{a}^{\top} & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{v}}_{\mathrm{N}} \\ \lambda_a \end{bmatrix} = \begin{bmatrix} -K\boldsymbol{u}_{\mathrm{N}} + \boldsymbol{f} \\ C(\boldsymbol{u}_{\mathrm{N}},\boldsymbol{v}_{\mathrm{N}}) \end{bmatrix} \tag{4.2.4}$$

where $C(\boldsymbol{u}_{\mathrm{N}},\boldsymbol{v}_{\mathrm{N}}) = 2\alpha\boldsymbol{a}^{\top}\boldsymbol{v}_{\mathrm{N}} + \alpha^2\boldsymbol{a}^{\top}\boldsymbol{u}_{\mathrm{N}}$. Note that the coefficient matrix of the above equation is constant. Given $\boldsymbol{u}_{\mathrm{N}}$ and $\boldsymbol{v}_{\mathrm{N}}$, we can calculate the right-side vector of the above equation, implying that solving the above linear equation yields $\dot{\boldsymbol{v}}_{\mathrm{N}}$. Consequently, given $\boldsymbol{u}_{\mathrm{N}}$ and $\boldsymbol{v}_{\mathrm{N}}$, we can calculate their time derivatives $\dot{\boldsymbol{u}}_{\mathrm{N}}$ and $\dot{\boldsymbol{v}}_{\mathrm{N}}$, which offers a canonical form of ordinary differential equations. Any ODE solver is available to solve the canonical form of ordinary differential equations numerically.



(a) body

(b) nodal point position

Figure 4.1: One-dimensional beam jumping

Example Let us calculate the dynamic deformation of an elastic beam of length L. Divide region $[\,0,\,L\,]$ into five small regions. During time interval $[\,0,\,t_{\rm push}\,]$, one end $\mathrm{P}(0)$ of the beam is in contact with the floor and a pushing force $f_{\rm push}$ is applied to the other end $\mathrm{P}(L)$ to shrink the beam. During $[\,t_{\rm push},\,t_{\rm end}\,]$, the applied force is released. A reaction force exerts to the contacting end as long as the end is in contact with the floor. Penalty method is applied to calculate the reaction force. Namely,

$$\text{reaction force} = \begin{cases} -k_{\rm floor}\,A\,u(0) & u(0) \leq 0 \\ 0 & u(0) > 0 \end{cases}$$

Figure 4.1 shows a calculation result with $L = 10\,\mathrm{cm}$, $A = 2\,\mathrm{cm}^2$, $E = 50\,\mathrm{kPa}$, $c = 0.2\,\mathrm{kPa}\cdot\mathrm{s}$, $\rho = 1.0\ \mathrm{g/cm}^3$, $f_{\rm push} = 2.0\,\mathrm{N}$, $t_{\rm push} = 0.2\,\mathrm{s}$, and $k_{\rm floor} = 0.1\,\mathrm{MPa/cm}$. Deformation of the beam during $[\,t_{\rm push},\,t_{\rm end}\,]$ and jumping motion during $[\,t_{\rm push},\,t_{\rm end}\,]$ are calculated properly.

4.3 Dynamic deformation of two-dimensional soft body

Let us formulate the dynamic deformation of a two-dimensional elastic body specified by S. Assume that Lamé's constants λ, μ, density ρ, and width h are uniform over the body, implying that they are constants. Approximating region S by a finite number of triangles and letting $\boldsymbol{u}_{\rm N}$ be nodal displacement vector, kinetic and strain potential energies are described as

$$T = \frac{1}{2}\dot{\boldsymbol{u}}_{\rm N}^{\top}\,M\,\dot{\boldsymbol{u}}_{\rm N}, \qquad U = \frac{1}{2}\boldsymbol{u}_{\rm N}^{\top}\,K\,\boldsymbol{u}_{\rm N}$$

where M denote the inertia matrix and K represent the stiffness matrix. Work done by external forces can be formulated as

$$W = \boldsymbol{f}^{\top}\boldsymbol{u}_{\rm N}$$

A set of geometric constraints imposed on the body can be described as

$$\boldsymbol{R} = A^{\top}\boldsymbol{u}_{\rm N} - \boldsymbol{b}(t) = \boldsymbol{0}$$

where $\boldsymbol{b}(t)$ is a collective vector specifying the position of constrained points at time t. Consequently, we have the following Lagrangian:

$$\begin{aligned}\mathcal{L}(\boldsymbol{u}_{\rm N}, \dot{\boldsymbol{u}}_{\rm N}) &= T - U + W + \boldsymbol{\lambda}^\top \boldsymbol{R} \\ &= \frac{1}{2}\dot{\boldsymbol{u}}_{\rm N}^\top M \,\dot{\boldsymbol{u}}_{\rm N} - \frac{1}{2}\boldsymbol{u}_{\rm N}^\top K \,\boldsymbol{u}_{\rm N} + \boldsymbol{f}^\top \boldsymbol{u}_{\rm N} + \boldsymbol{\lambda}^\top (A^\top \boldsymbol{u}_{\rm N} - \boldsymbol{b}(t)),\end{aligned} \tag{4.3.1}$$

where $\boldsymbol{\lambda}$ is a collective vector of Langrange multipliers corresponding to a set of constraints. Since M and K are constant matrices, we have

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}_{\rm N}} = -K\boldsymbol{u}_{\rm N} + \boldsymbol{f} + A\boldsymbol{\lambda}, \qquad \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}_{\rm N}} = M\dot{\boldsymbol{u}}_{\rm N}$$

which directly yields

$$-K\boldsymbol{u}_{\rm N} + \boldsymbol{f} + A\boldsymbol{\lambda} - M\ddot{\boldsymbol{u}}_{\rm N} = \boldsymbol{0} \tag{4.3.2}$$

Equation for stabilizing constraint $A\boldsymbol{u}_{\rm N} = \boldsymbol{0}$ is given by

$$(A^\top \ddot{\boldsymbol{u}}_{\rm N} - \ddot{\boldsymbol{b}}(t)) + 2\alpha(A^\top \dot{\boldsymbol{u}}_{\rm N} - \dot{\boldsymbol{b}}(t)) + \alpha^2 (A^\top \boldsymbol{u}_{\rm N} - \boldsymbol{b}(t)) = \boldsymbol{0}, \tag{4.3.3}$$

where α is a positive constant. Introducing $\boldsymbol{v}_{\rm N} = \dot{\boldsymbol{u}}_{\rm N}$, the above two ordinary differential equations collectively turn into

$$\begin{bmatrix} M & -A \\ -A^\top & \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{v}}_{\rm N} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} -K\boldsymbol{u}_{\rm N} + \boldsymbol{f} \\ C(\boldsymbol{u}_{\rm N}, \boldsymbol{v}_{\rm N}) \end{bmatrix} \tag{4.3.4}$$

where

$$\boldsymbol{C}(\boldsymbol{u}_{\rm N}, \boldsymbol{v}_{\rm N}) = -\ddot{\boldsymbol{b}}(t) + 2\alpha(A^\top \boldsymbol{v}_{\rm N} - \dot{\boldsymbol{b}}(t)) + \alpha^2(A^\top \boldsymbol{u}_{\rm N} - \boldsymbol{b}(t)).$$

The above equation provides a canonical form of ordinary differential equations, which can be solved numerically by any ODE solver.

Damping forces Let us introduce damping forces caused by viscosity of robot body material. Let c be *viscous modulus* of the material. Linear isotropic viscosity can be characterized by two constants

$$\lambda^{vis} = \frac{\nu c}{(1+\nu)(1-2\nu)}, \qquad \mu^{vis} = \frac{c}{2(1+\nu)},$$

where ν denote Poisson's ratio. Then, a set of damping forces at nodal points is described by

$$-B\dot{\boldsymbol{u}}_{\rm N}$$

where

$$B = \lambda^{vis} J_\lambda + \mu^{vis} J_\mu$$

is referred to as *damping matrix*. Replacing elastic forces $-K\boldsymbol{u}_{\rm N}$ in eq. (4.3.4) by viscoelastic forces $-K\boldsymbol{u}_{\rm N} - B\boldsymbol{v}_{\rm N}$, we can compute the dynamic deformation of a viscoelastic body.

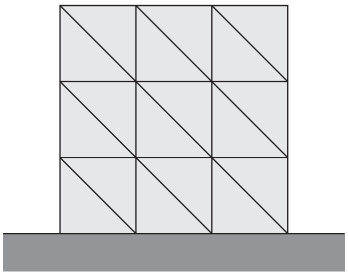


Figure 4.2: Square body on a floor

Example (deforming body) Let us calculate the dynamic deformation of a two-dimensional elastic square body of width w shown in Fig. 4.2. Let us divide the square region into $3\times3\times2$ triangles. During time interval $[\,0,\, t_{\text{push}}\,]$, the bottom of the body is fixed to the floor and edge $\text{P}_{14}\text{P}_{15}$ moves downward at a constant velocity v_{push}. During $[\,t_{\text{push}},\, t_{\text{hold}}\,]$, the bottom remains fixed and and edge $\text{P}_{14}\text{P}_{15}$ keeps its position. During $[\,t_{\text{hold}},\, t_{\text{end}}\,]$, the bottom remains fixed while $\text{P}_{14}\text{P}_{15}$ is released.

During $[\,0,\, t_{\text{push}}\,]$, the following constraints are imposed on the square body:

$$
\begin{aligned}
&\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}_3 = \boldsymbol{u}_4 = \boldsymbol{0} \\
&\boldsymbol{u}_{14} = \boldsymbol{u}_{15} = \boldsymbol{0} + \boldsymbol{v}_{\text{push}} t
\end{aligned}
$$

where $\boldsymbol{v}_{\text{push}} = [\,0,\, -v_{\text{push}}\,]^\top$. Matrix

$$
A^\top = \begin{array}{c}
\begin{bmatrix}
I & & & & \cdots & & \\
& I & & & \cdots & & \\
& & I & & \cdots & & \\
& & & I & \cdots & & \\
& & & & \cdots & I & \\
& & & & \cdots & & I
\end{bmatrix} \\
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & & 14 & \text{15-th block columns} \end{array}
\end{array}
$$

specifies the nodal point displacements under constraints, that is,

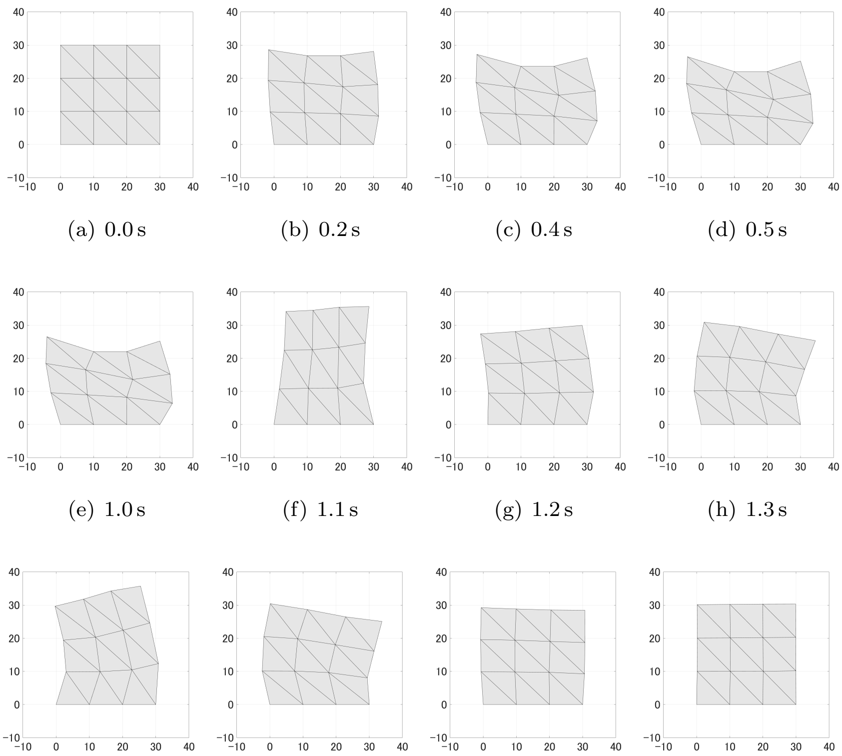
$$
A^\top \boldsymbol{u}_{\text{N}} = \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \boldsymbol{u}_4 \\ \boldsymbol{u}_{14} \\ \boldsymbol{u}_{15} \end{bmatrix}
$$

A collective vector specifying the position of constrained points at time t is then given by

$$
\boldsymbol{b}(t) = \boldsymbol{b}_0 + \boldsymbol{b}_1 t
$$

where

$$
\boldsymbol{b}_0 = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}, \qquad
\boldsymbol{b}_1 = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{v}_{\text{push}} \\ \boldsymbol{v}_{\text{push}} \end{bmatrix}
$$



(a) 0.0 s (b) 0.2 s (c) 0.4 s (d) 0.5 s

(e) 1.0 s (f) 1.1 s (g) 1.2 s (h) 1.3 s

(i) 1.4 s (j) 1.5 s (k) 2.0 s (l) 3.0 s

Figure 4.3: Dynamic deformation of an elastic square body ($3 \times 3 \times 2$ triangles)

Noting that $\dot{\boldsymbol{b}}(t) = \boldsymbol{b}_1$ and $\ddot{\boldsymbol{b}}(t) = \boldsymbol{0}$, we find

$$\boldsymbol{C}(\boldsymbol{u}_{\rm N}, \boldsymbol{v}_{\rm N}) = 2\alpha(A^\top \boldsymbol{v}_{\rm N} - \boldsymbol{b}_1) + \alpha^2(A^\top \boldsymbol{u}_{\rm N} - (\boldsymbol{b}_0 + \boldsymbol{b}_1 t)).$$

During $[\, t_{\rm push},\, t_{\rm hold}\,]$, we find

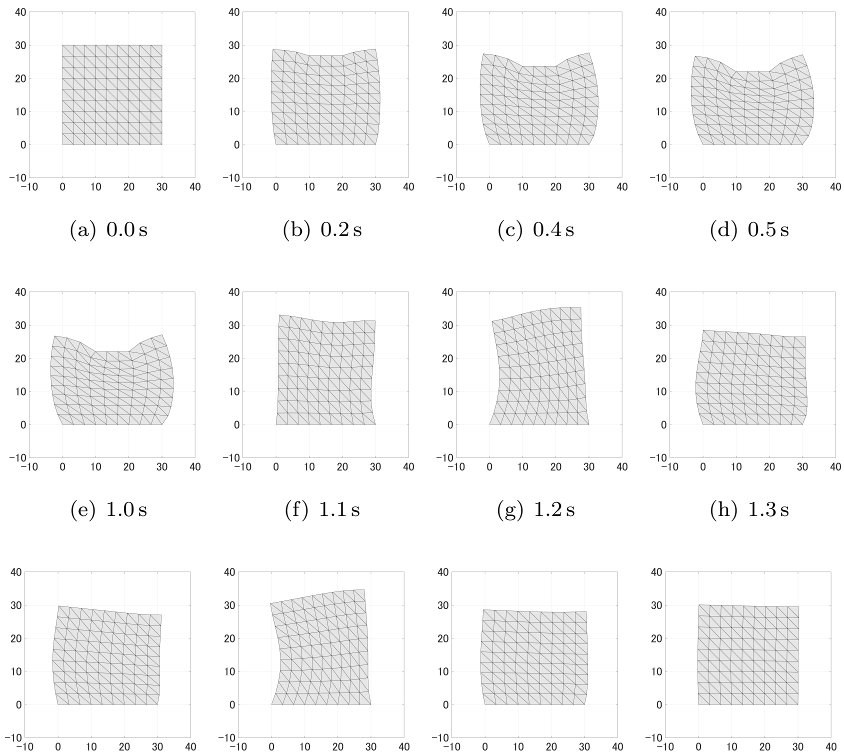
$$\boldsymbol{b}_0 = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{v}_{\rm push} t_{\rm push} \\ \boldsymbol{v}_{\rm push} t_{\rm push} \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$$

During $[\, t_{\rm hold},\, t_{\rm end}\,]$, we have the following constraints:

$$\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}_3 = \boldsymbol{u}_4 = \boldsymbol{0},$$

which yields,

$$A^\top = \begin{bmatrix} I & & & & \cdots \\ & I & & & \cdots \\ & & I & & \cdots \\ & & & I & \cdots \end{bmatrix}$$



(a) 0.0 s (b) 0.2 s (c) 0.4 s (d) 0.5 s

(e) 1.0 s (f) 1.1 s (g) 1.2 s (h) 1.3 s

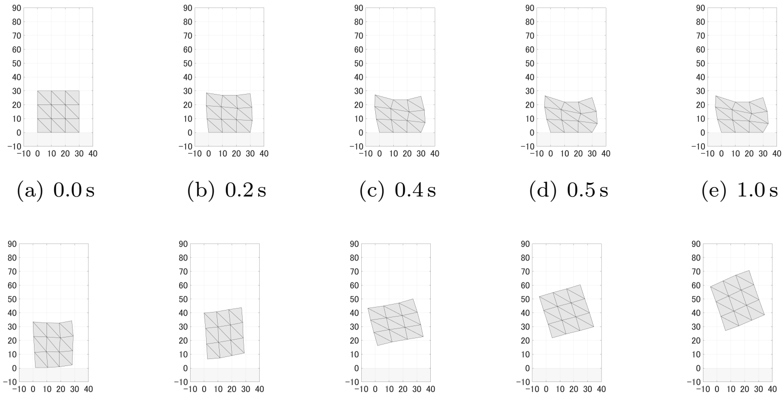
(i) 1.4 s (j) 1.5 s (k) 2.0 s (l) 3.0 s

Figure 4.4: Dynamic deformation of an elastic square body ($9 \times 9 \times 2$ triangles)

$$\boldsymbol{b}_0 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{b}_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

Figure 4.3 shows a snapshot of the computation result with $w = 30\,\mathrm{cm}$, $h = 1\,\mathrm{cm}$, $E = 1.0\,\mathrm{MPa}$, $c = 40\,\mathrm{Pa}\cdot\mathrm{s}$, $\nu = 0.48$, $\rho = 1.0\ \mathrm{g/cm^3}$, $t_{\mathrm{push}} = 0.5\,\mathrm{s}$, $t_{\mathrm{hold}} = 1.0\,\mathrm{s}$, and $v_{\mathrm{push}} = 16\,\mathrm{cm/s}$. Figure 4.4 shows a snapshot of the computation result under a finer mesh; the square region consists of $9 \times 9 \times 2$ triangles. These computation results demonstrate that deformation can be simulated properly. Additionally, computation results depend on mesh.

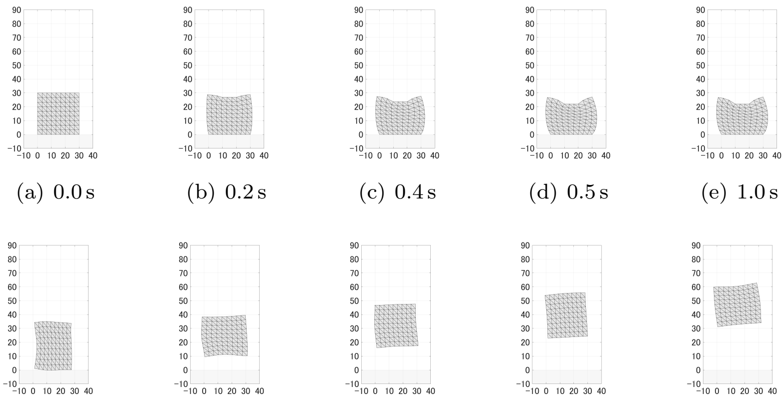
Penalty method When an elastic body contacts with an obstacle, reaction forces are applied to nodal points of the body. Penalty method provides a formulation of reaction forces causes by the contact. In penalty method, nodal points may interfere with obstacles, but penalty forces are applied to interfering nodal points to dissolve the interference. Let us introduce a signed distance $d(\boldsymbol{x})$ to describe an obstacle. Function $d(\boldsymbol{x})$ is equal to 0 at any point on the surface of the obstacle, negative inside the obstacle, and positive outside the obstacle. Gradient vector $\nabla g = [\,\partial g/\partial x,\, \partial g/\partial y\,]^\top$ is the outside normal vector of its magnitude 1. We apply a penalty force to a nodal point inside an obstacle to avoid mechanical interference between the point and the obstacle. Assume that the penalty force is given by virtual spring and damper between an interfering nodal point and an obstacle.



(a) 0.0 s (b) 0.2 s (c) 0.4 s (d) 0.5 s (e) 1.0 s

(f) 1.1 s (g) 1.2 s (h) 1.3 s (i) 1.4 s (j) 1.5 s

Figure 4.5: Jumping of an elastic square body ($3 \times 3 \times 2$ triangles)



(a) 0.0 s (b) 0.2 s (c) 0.4 s (d) 0.5 s (e) 1.0 s

(f) 1.1 s (g) 1.2 s (h) 1.3 s (i) 1.4 s (j) 1.5 s

Figure 4.6: Jumping of an elastic square body ($9 \times 9 \times 2$ triangles)

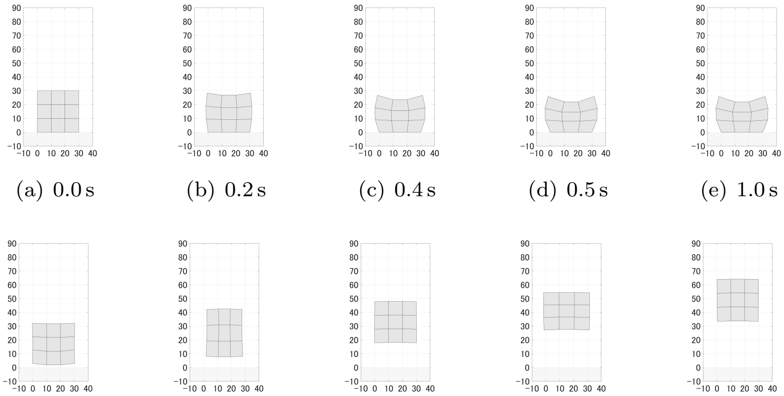
Noting that the direction of the penalty force is given by gradient vector ∇g, the penalty force is formulated as

$$\boldsymbol{f}_{\rm p} = \begin{cases} (-K_{\rm p} d - B_{\rm p} \dot{d}) \, \nabla d & d(\boldsymbol{x}) \leq 0 \\ \boldsymbol{0} & \text{otherwise} \end{cases} \tag{4.3.5}$$

where $K_{\rm p}$ denotes contact stiffness coefficient and $B_{\rm p}$ represents contact damping coefficient.

For example, assume that a floor is described as $d(\boldsymbol{x}) = y$, that is, the floor region is given by $y \leq 0$ and the floor surface coincides with x-axis. Gradient vector is given by $\nabla g = [\,0, 1\,]^{\top}$. Noting that position of k-th nodal point P_k is given by $[\,x_k + u_k,\, y_k + v_k\,]^{\top}$, where x_k and y_k are independent of time, the penalty force applied to P_k is described as

$$\boldsymbol{f}_{\rm p} = \begin{cases} \begin{bmatrix} 0 \\ -K_{\rm p}(y_k + v_k) - B_{\rm p}\dot{v}_k \end{bmatrix} & y_k + v_k \leq 0 \\ \boldsymbol{0} & \text{otherwise} \end{cases} \tag{4.3.6}$$



(a) 0.0 s (b) 0.2 s (c) 0.4 s (d) 0.5 s (e) 1.0 s

(f) 1.1 s (g) 1.2 s (h) 1.3 s (i) 1.4 s (j) 1.5 s

Figure 4.7: Jumping of an elastic square body (3×3 rectangular elements)

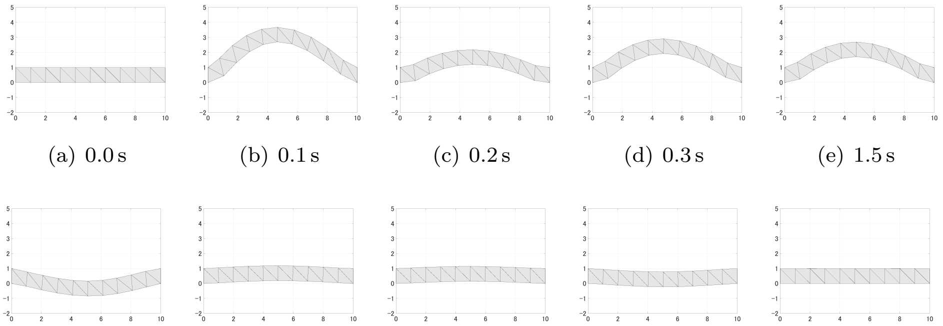
The above equation provides reaction forces applied to an elastic body from the floor.

Example (jumping body) Let us simulate the jumping of a two-dimensional elastic square body. Divide the square region into $3 \times 3 \times 2$ triangles (Fig. 4.2). During $[\,0,\, t_{\rm push}\,]$, the bottom of the body is fixed to the floor and edge $\mathrm{P}_{14}\mathrm{P}_{15}$ moves downward at a constant velocity $v_{\rm push}$. During $[\,t_{\rm push},\, t_{\rm hold}\,]$, the bottom remains fixed and and edge $\mathrm{P}_{14}\mathrm{P}_{15}$ keeps its position. During $[\,t_{\rm hold},\, t_{\rm end}\,]$, all constraints are released, but reaction forces are applied to the body from the floor. We apply penalty method to calculate reaction forces. Figure 4.5 shows a snapshot of the computation result with width $w = 30\,\mathrm{cm}$, thickness $h = 1\,\mathrm{cm}$, Young's modulus $E = 1.0\,\mathrm{MPa}$, viscous modulus $c = 40\,\mathrm{Pa}\cdot\mathrm{s}$, Poisson's ratio $\nu = 0.48$, density $\rho = 1.0\ \mathrm{g/cm^3}$, $t_{\rm push} = 0.5\,\mathrm{s}$, $t_{\rm hold} = 1.0\,\mathrm{s}$, $v_{\rm push} = 16\,\mathrm{cm/s}$, contact stiffness coefficient $K_{\rm p} = 100\ \mathrm{N/\,m}$, and contact damping coefficient $B_{\rm p} = 0\ \mathrm{N/(\,m/\,s)}$. The computed result demonstrates the jumping of an elastic body but the body rotates during jumping, which originates from asymmetric triangular mesh. Figure 4.6 shows a snapshot of the computation result under a finer mesh; the square region consists of $9 \times 9 \times 2$ triangles. In this computation, contact stiffness coefficient is $K_{\rm p} = 100\ \mathrm{N/\,m} \times (4/10)$, since the bottom of the body includes 10 nodal points whereas 4 nodal points in Figure 4.5. Comparing Figs. 4.5 and 4.6, we find computation results depend on mesh and finer mesh yields better result.

Example (rectangular elements) Rectangular elements can be applied to computation of dynamic motion and deformation. Recall that stiffness and inertia matrices of an elastic body consisting of rectangular elements are formulated in eqs. (3.6.6)(3.6.10) (see Section 3.6). Applying these matrices to eq. (4.3.4), we can calculate the dynamic motion and deformation of the body.

Let us simulate the jumping of a two-dimensional elastic square body. Divide the square region into 3×3 rectangular regions. Figure 4.7 shows a snapshot of the computation result. Note that results based on triangular elements (Figs. 4.5 and 4.6) lost symmetry in deformation and the body rotated during jumping while the result based on rectangular elements kept symmetry in deformation and little rotation happened during jumping.

Nodal force vector caused by pressure Pressure applied into a chamber causes distributed stress along the boundary of the chamber area. Assuming that the increment of the



(a) 0.0 s (b) 0.1 s (c) 0.2 s (d) 0.3 s (e) 1.5 s

(f) 1.6 s (g) 1.7 s (h) 1.8 s (i) 1.9 s (j) 3.0 s

Figure 4.8: Dynamic expansion of an elastic membrane by air pressure

chamber area is described by a polygon, distributed stress turns into a set of nodal forces at vertices of the polygon. Let $S(\boldsymbol{u}_{\rm N})$ be the area of the polygon and p be the applied pressure at time t. Since work done by the applied pressure is given by $ph\, S(\boldsymbol{u}_{\rm N})$, nodal force vector is described as $ph\, \partial S/\partial \boldsymbol{u}_{\rm N}$. Then, a set of Lagrange equations of motion and deformation is described as follows:

$$M\dot{\boldsymbol{v}}_{\rm N} - A\boldsymbol{\lambda} = -K\boldsymbol{u}_{\rm N} - B\boldsymbol{v}_{\rm N} + ph\frac{\partial S}{\partial \boldsymbol{u}_{\rm N}}$$

Note that gradient vector $\partial S/\partial \boldsymbol{u}_{\rm N}$ can be calculated from geometry of the polygon (see Problem 6 in Section 3). Method **surrounded_area_gradient** of class `Body` calculates this gradient vector.

Let us compute dynamic deformation of an elastic membrane shown in Fig. 3.5. Membrane material is characterized by Young's modulus $E = 0.1\,\mathrm{MPa}$, viscous modulus $c = 40\,\mathrm{Pa}\cdot\mathrm{s}$, Poisson's ratio $\nu = 0.48$, and density $\rho = 1.0$ g/cm^3. Pressure at time t is given by

$$p(t) = \begin{cases} 2.0\,\mathrm{kPa} & t \le 1.5\,\mathrm{s} \\ 0.0\,\mathrm{kPa} & t > 1.5\,\mathrm{s} \end{cases}$$

Figure 4.8 shows a snapshot of the computation result. The membrane expands outward while positive pressure is applied, then recovers its natural shape after the pressure is released.

4.4 Nodal point forces based on Green strain

Simulation of dynamic motion and deformation requires nodal point forces. When we apply Cauchy strain, a set of nodal point forces is calculated by a linear description $-K\boldsymbol{u}_{\rm N}$, since strain potential energy based on Cauchy strain is quadratic with respect to nodal point displacements. When we apply Green strain, strain potential energy turns quartic with respect to nodal point displacements, suggesting that nodal point forces are cubic with respect to the displacements. Instead of formulating complex cubic description, we construct a procedure to calculate nodal point forces based on Green strain.

Let us formulate nodal point forces caused by Green strain $\boldsymbol{E}$ at triangle $\mathrm{T}_p = \triangle \mathrm{P}_i\mathrm{P}_j\mathrm{P}_k$. Partial strain potential energy stored in triangle T_p is described as

$$U_p = \frac{1}{2}\boldsymbol{E}^{\top}(\lambda I_\lambda + \mu I_\mu)\boldsymbol{E}\,\triangle h$$

where $\triangle = \triangle \mathrm{P}_i \mathrm{P}_j \mathrm{P}_k$. Differentiating the above equation with respect to $\boldsymbol{\gamma}_u$ and $\boldsymbol{\gamma}_v$, we find

$$\frac{\partial U_p}{\partial \boldsymbol{\gamma}_u} = \frac{\partial \boldsymbol{E}^\top}{\partial \boldsymbol{\gamma}_u} \frac{\partial U_p}{\partial \boldsymbol{E}}, \qquad \frac{\partial U_p}{\partial \boldsymbol{\gamma}_v} = \frac{\partial \boldsymbol{E}^\top}{\partial \boldsymbol{\gamma}_v} \frac{\partial U_p}{\partial \boldsymbol{E}}$$

where

$$\frac{\partial U_p}{\partial \boldsymbol{E}} = (\lambda I_\lambda + \mu I_\mu) \triangle h \, \boldsymbol{E}$$

and

$$\frac{\partial \boldsymbol{E}^\top}{\partial \boldsymbol{\gamma}_u} = \left[\begin{array}{ccc} \dfrac{\partial E_{xx}}{\partial \boldsymbol{\gamma}_u} & \dfrac{\partial E_{yy}}{\partial \boldsymbol{\gamma}_u} & \dfrac{\partial (2E_{xy})}{\partial \boldsymbol{\gamma}_u} \end{array} \right], \qquad \frac{\partial \boldsymbol{E}^\top}{\partial \boldsymbol{\gamma}_v} = \left[\begin{array}{ccc} \dfrac{\partial E_{xx}}{\partial \boldsymbol{\gamma}_v} & \dfrac{\partial E_{yy}}{\partial \boldsymbol{\gamma}_v} & \dfrac{\partial (2E_{xy})}{\partial \boldsymbol{\gamma}_v} \end{array} \right]$$

Calculating the above partial derivatives, we have

$$\begin{aligned}
\frac{\partial E_{xx}}{\partial \boldsymbol{\gamma}_u} &= \boldsymbol{a} + (\boldsymbol{a}^\top \boldsymbol{\gamma}_u)\boldsymbol{a} &&= (1 + u_x)\boldsymbol{a} \\
\frac{\partial E_{yy}}{\partial \boldsymbol{\gamma}_u} &= (\boldsymbol{b}^\top \boldsymbol{\gamma}_u)\boldsymbol{b} &&= u_y \boldsymbol{b} \\
\frac{\partial (2E_{xy})}{\partial \boldsymbol{\gamma}_v} &= \boldsymbol{b} + \boldsymbol{a}(\boldsymbol{b}^\top \boldsymbol{\gamma}_u) + (\boldsymbol{a}^\top \boldsymbol{\gamma}_u)\boldsymbol{b} &&= (1 + u_x)\boldsymbol{b} + u_y \boldsymbol{a} \\
\frac{\partial E_{xx}}{\partial \boldsymbol{\gamma}_v} &= (\boldsymbol{a}^\top \boldsymbol{\gamma}_v)\boldsymbol{a} &&= v_x \boldsymbol{a} \\
\frac{\partial E_{yy}}{\partial \boldsymbol{\gamma}_v} &= \boldsymbol{b} + (\boldsymbol{b}^\top \boldsymbol{\gamma}_v)\boldsymbol{b} &&= (1 + v_y)\boldsymbol{b} \\
\frac{\partial (2E_{xy})}{\partial \boldsymbol{\gamma}_v} &= \boldsymbol{a} + \boldsymbol{a}(\boldsymbol{b}^\top \boldsymbol{\gamma}_v) + (\boldsymbol{a}^\top \boldsymbol{\gamma}_v)\boldsymbol{b} &&= (1 + v_y)\boldsymbol{a} + v_x \boldsymbol{b}
\end{aligned}$$

Consequently,

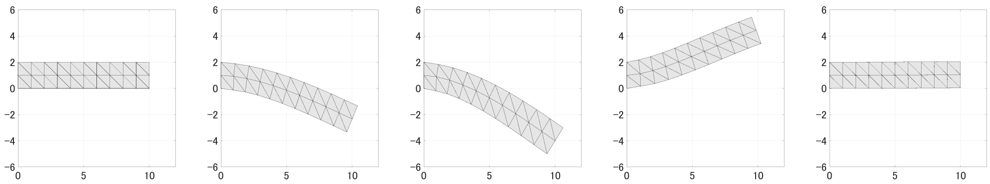
$$\begin{aligned}
\frac{\partial U_p}{\partial \boldsymbol{\gamma}_u} &= \left[\begin{array}{ccc} (1 + u_x)\boldsymbol{a} & u_y \boldsymbol{b} & (1 + u_x)\boldsymbol{b} + u_y \boldsymbol{a} \end{array} \right] \frac{\partial U_p}{\partial \boldsymbol{E}} \\
\frac{\partial U_p}{\partial \boldsymbol{\gamma}_v} &= \left[\begin{array}{ccc} v_x \boldsymbol{a} & (1 + v_y)\boldsymbol{b} & (1 + v_y)\boldsymbol{a} + v_x \boldsymbol{b} \end{array} \right] \frac{\partial U_p}{\partial \boldsymbol{E}}
\end{aligned}$$

From the above two partial derivatives, we obtain partial derivatives $\partial U_p / \partial \boldsymbol{u}_i$, $\partial U_p / \partial \boldsymbol{u}_j$, and $\partial U_p / \partial \boldsymbol{u}_k$. Namely, the first elements of the above two partial derivatives yield $\partial U_p / \partial \boldsymbol{u}_i$, their second elements yield $\partial U_p / \partial \boldsymbol{u}_j$, and their third elements yield $\partial U_p / \partial \boldsymbol{u}_k$. Nodal point forces at P_i, P_j, P_k caused by partial strain potential energy U_p are then formulated as

$$\boldsymbol{f}_i^p = -\frac{\partial U_p}{\partial \boldsymbol{u}_i}, \qquad \boldsymbol{f}_j^p = -\frac{\partial U_p}{\partial \boldsymbol{u}_j}, \qquad \boldsymbol{f}_k^p = -\frac{\partial U_p}{\partial \boldsymbol{u}_k}$$

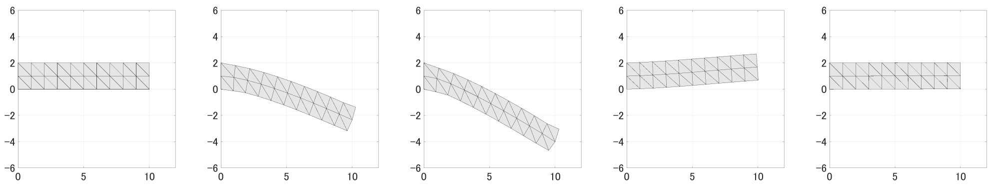
Synthesizing nodal point forces of all triangles, we obtain nodal force vector caused by Green strain, that is,

$$\boldsymbol{f}_\mathrm{N} = \bigoplus_p \left[\begin{array}{c} \boldsymbol{f}_i^p \\ \boldsymbol{f}_j^p \\ \boldsymbol{f}_k^p \end{array} \right] \tag{4.4.1}$$



(a) 0.0 s (b) 0.2 s (c) 0.5 s (d) 0.6 s (e) 2.0 s

Figure 4.9: Dynamic bending of elastic bending (Cauchy strain)



(a) 0.0 s (b) 0.2 s (c) 0.5 s (d) 0.6 s (e) 2.0 s

Figure 4.10: Dynamic bending of elastic bending (Green strain)

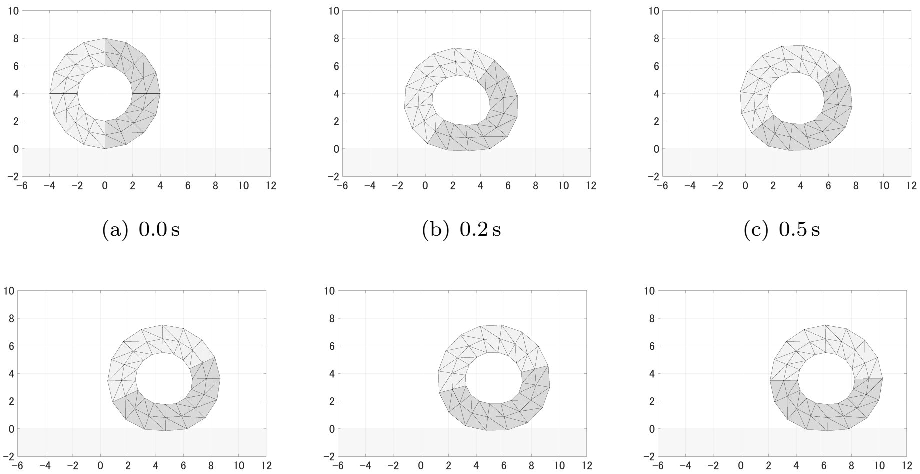
For example, nodal force vector of a rectangle region shown in Fig. 2.2 is described as follows:

$$\boldsymbol{f}_{\mathrm{N}} = \begin{bmatrix} \boldsymbol{f}_1^1 & & & & & & \\ \boldsymbol{f}_2^1 & + & \boldsymbol{f}_2^2 & + & \boldsymbol{f}_2^3 & & \\ & & \boldsymbol{f}_3^2 & & & + & \boldsymbol{f}_3^4 \\ \boldsymbol{f}_4^1 & & & + & \boldsymbol{f}_4^3 & & \\ & & \boldsymbol{f}_5^2 & + & \boldsymbol{f}_5^3 & + & \boldsymbol{f}_5^4 \\ & & & & & & \boldsymbol{f}_6^4 \end{bmatrix}$$

Replacing $-K\boldsymbol{u}_{\mathrm{N}}$ in eq. (4.3.4) by the above nodal force vector $\boldsymbol{f}_{\mathrm{N}}$, we obtain dynamic equation of deformation based on Green strain. Method **nodal_forces_Green_strain** of class `Body` calculates nodal force vector based on Green strain for given nodal point displacements.

The above procedure is able to calculate Green strain based nodal force vector, but requires much calculation time. Section 4.5 details an improved procedure to calculate Green strain based nodal force vector through multiplication of coefficient matrices and collective vectors. Before calculating Green strain based nodal force vector, method **calculate_coefficient_matrices_for_Green_strain** of class `Body` should be performed once to prepare coefficient matrices for the calculation.

Example (beam bending) Let us simulate the dynamic bending of an elastic beam of its length 10 cm, height 2 cm, and thickness 1 cm. One end of the beam is fixed to a wall while the center of the other end is push downward by 5 cm during 5 s, before releasing the pushing constraint. Assume the material of the beam exhibits isotropic linear elasticity, specified by $E = 0.1\,\mathrm{MPa}$, $c = 4.0\,\mathrm{Pa}\cdot\mathrm{s}$, and $\nu = 0.48$. Figure 4.9 shows the computation based on Cauchy strain. Dynamic deformation is formulated as eq. (4.3.4). The right end of the beam unnaturally expands in the deformed shape. Figure 4.10 shows the computation based on Green strain. We find that the beam bends naturally, avoiding the unnatural expansion of the elements.



(a) 0.0 s (b) 0.2 s (c) 0.5 s

(d) 1.0 s (e) 2.0 s (f) 4.0 s

Figure 4.11: Rolling of elastic ring

Example (rolling contact) Let us calculate dynamic behavior of an elastic ring rolling on a flat floor. We focus on the two-dimensional deformation of cross-sectional area of the ring. The outer and inner radii of the ring are 4 mm and 2 mm in its natural shape (Fig. 4.11(a)). Material of the ring exhibits isotropic linear elasticity, specified by $E = 0.1\,\mathrm{MPa}$ and $\nu = 0.48$. Green strain is used during the calculation. Viscous modulus of the material is $c = 0.04\,\mathrm{kPa{\cdot}s}$. Density of the material is $\rho_{\mathrm{left}} = 1\ \mathrm{g/cm^3}$ in the left half of the ring while $\rho_{\mathrm{right}} = 10\ \mathrm{g/cm^3}$ in the right half. Gravitational force acts along vertical direction downward with acceleration of gravity $g = 9.8\,\mathrm{m/s^2}$. Difference in density causes the ring rolling clockwise.

Let us extend penalty method eq. (4.3.5) by introducing tangential damping so as to simulate the rolling on a floor. Let $d(\boldsymbol{x})$ be the signed distance specifying the floor. Velocity $\dot{\boldsymbol{x}}$ can be decomposed into normal component $(\nabla g\, \nabla g^{\top})\, \dot{\boldsymbol{x}}$ and tangential component $(I - \nabla g\, \nabla g^{\top})\, \dot{\boldsymbol{x}}$. Assuming that damping force acts along tangential direction, penalty force is described as

$$\boldsymbol{f}_{\mathrm{p}} = \begin{cases} (-K_{\mathrm{p}} d - B_{\mathrm{p}} \dot{d})\, \nabla d - B_{\mathrm{t}} (I - \nabla g\, \nabla g^{\top})\, \dot{\boldsymbol{x}} & d(\boldsymbol{x}) \leq 0 \\ \boldsymbol{0} & \text{otherwise} \end{cases} \tag{4.4.2}$$

where B_{t} represents tangential damping coefficient. Especially, when a floor is specified by $d(\boldsymbol{x}) = y$, that is, the floor region is given by $y \leq 0$, penalty force applied to nodal point P_k is described as

$$\boldsymbol{f}_{\mathrm{p}} = \begin{cases} \begin{bmatrix} -B_{\mathrm{t}} \dot{u}_k \\ -K_{\mathrm{p}} (y_k + v_k) - B_{\mathrm{p}} \dot{v}_k \end{bmatrix} & y_k + v_k \leq 0 \\ \boldsymbol{0} & \text{otherwise} \end{cases} \tag{4.4.3}$$

The above equation provides reaction forces applied to an elastic body from the floor.

Figure 4.11 shows a snapshot of a simulation result with penalty method parameters $K_{\mathrm{p}} = 1000\ \mathrm{N/m}$, $B_{\mathrm{p}} = 0\ \mathrm{N/(m/s)}$, and $B_{\mathrm{t}} = 20\ \mathrm{N/(m/s)}$. The elastic ring deforms under gravity, rotating clockwise and moving rightward (Fig. 4.11(b) through Fig. 4.11(e)). The ring turns stationary when heavier half comes downward (Fig. 4.11(f)).

Example (elastic ring with eight spring actuators) Let us calculate dynamic behavior of an elastic ring driven by eight spring actuators. We focus on the two-dimensional deformation of cross-sectional area of the ring (see Fig. 3.11(a) in Section 3.5). The outer and inner radii of the ring are 5 mm and 4 mm in its natural shape. Material of the ring exhibits isotropic linear elasticity, specified by $E = 0.1\,\mathrm{MPa}$ and $\nu = 0.48$. Green strain is used during the calculation. Let density and viscous modulus of ring material be $\rho = 1\ \mathrm{g/cm^3}$ and $c = 0.04\,\mathrm{kPa}\cdot\mathrm{s}$. Eight spring actuators labeled A_1 through A_8 counterclockwise are radially distributed inside the ring. Mass m supports the spring actuators, that is, one end of each spring actuator is connected to the mass and the other end is connected to inner surface of the ring. In natural state, actuator A_1 is below the mass. Let mass be $m = 1\,\mathrm{g}$ and acceleration of gravity be $g = 9.8\,\mathrm{m/s^2}$. Assume that natural length of all actuators be $L = 4\,\mathrm{mm}$, that is, the natural length is equal to the inner radius of the ring. Let spring constant of all actuators be $k = 20\ \mathrm{N/m}$.

Note that the total Lagrangian is the sum of Lagrangian of the elastic ring and Lagrangian of of the mass and spring actuators. Lagrangian of the elastic ring yields dynamic equation of motion and deformation of the elastic ring alone. We formulate Lagrangian of the mass and spring actuators to formulate the dynamic equation of the mass as well as additional forces applied to the elastic ring through spring actuators. Letting $\boldsymbol{x}_{\mathrm{mass}}$ be positional vector of the mass, Lagrangian of the mass and spring actuators is formulated as

$$\mathcal{L}' = \frac{1}{2} m\, \dot{\boldsymbol{x}}_{\mathrm{mass}}^{\top}\, \dot{\boldsymbol{x}}_{\mathrm{mass}} - \left(U_{\mathrm{springs}} - m\boldsymbol{g}^{\top}\boldsymbol{x}_{\mathrm{mass}}\right) + W$$

where $\boldsymbol{g} = [\,0,\ -g\,]^{\top}$ represents gravitational acceleration vector. Letting

$$\boldsymbol{e}_i = \frac{\boldsymbol{x}_{\mathrm{mass}} - (\boldsymbol{x}_j + \boldsymbol{u}_j)}{\|\boldsymbol{x}_{\mathrm{mass}} - (\boldsymbol{x}_j + \boldsymbol{u}_j)\|}$$

we have $\partial d_i/\partial \boldsymbol{x}_{\mathrm{mass}} = \boldsymbol{e}_i$ and $\partial d_i/\partial \boldsymbol{u}_j = -\boldsymbol{e}_i$ (see Problem 1), which directly yield

$$\frac{\partial}{\partial \boldsymbol{x}_{\mathrm{mass}}}\left(\frac{1}{2}kd_i^2\right) = kd_i\boldsymbol{e}_i, \quad \frac{\partial}{\partial \boldsymbol{u}_j}\left(\frac{1}{2}kd_i^2\right) = -kd_i\boldsymbol{e}_i$$

Lagrange equation of motion of mass m is described as

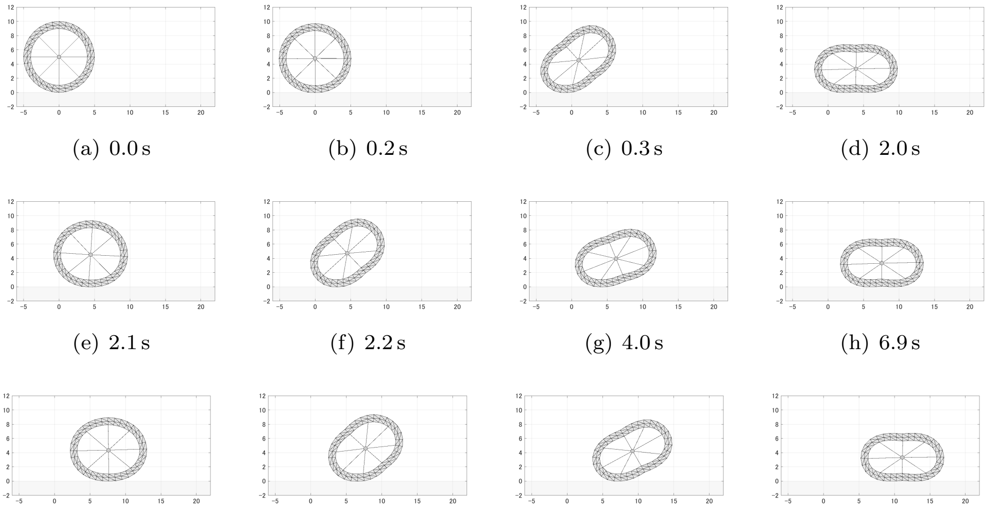
$$m\ddot{\boldsymbol{x}}_{\mathrm{mass}} = \sum_{i=1}^{8}(f_i - kd_i)\boldsymbol{e}_i + m\boldsymbol{g}$$

The following additional force is applied to nodal point P_j:

$$\frac{\partial \mathcal{L}'}{\partial \boldsymbol{u}_j} = -(f_i - kd_i)\boldsymbol{e}_i$$

Solving dynamic equations of motion and deformation with respect to $\boldsymbol{u}_{\mathrm{N}}$ and $\boldsymbol{x}_{\mathrm{mass}}$, we obtain the behavior of an elastic ring driven by spring actuators.

Let us simulate dynamic rolling of the ring on a floor. Rolling on the floor is described by penalty method with tangential damping. Penalty method parameters are $K_{\mathrm{p}} = 1000\ \mathrm{N/m}$, $B_{\mathrm{p}} = 0\ \mathrm{N/(m/s)}$, and $B_{\mathrm{t}} = 20\ \mathrm{N/(m/s)}$. Figure 4.12 shows a snapshot of a computation result. The natural shape (Fig. 4.12(a)) of the elastic ring deforms under gravity for 0.2 s (Fig. 4.12(b)). A pair of opposite spring actuators A_2 and A_6 apply shrinking force of 5 N for 1.8 s (Fig. 4.12(b) through (Fig. 4.12(d)). After relaxing all spring actuators for 0.1 s (Fig. 4.12(d) through Fig. 4.12(e)), another pair of opposite spring actuators A_3 and A_7



(a) 0.0 s (b) 0.2 s (c) 0.3 s (d) 2.0 s

(e) 2.1 s (f) 2.2 s (g) 4.0 s (h) 6.9 s

(i) 7.0 s (j) 7.1 s (k) 9.0 s (l) 12.0 s

Figure 4.12: Rolling of elastic ring driven by spring actuators

apply shrinking force of 5 N for 4.8 s (Fig. 4.12(e) through (Fig. 4.12(h)). After relaxing all spring actuators for 0.1 s (Fig. 4.12(h) through (Fig. 4.12(i)), a pair of opposite spring actuators A_4 and A_8 apply shrinking force of 5 N (Fig. 4.12(i) through (Fig. 4.12(l)). This result demonstrates the elastic ring rolls rightward through its deformation caused by successive activation of spring actuators: A_2 and A_6, A_3 and A_7, then A_4 and A_8.

4.5 Direct computation of Green strain based forces

Here we establish a procedure to directly compute nodal point forces based on Green strain. Note that Green strain components are quadratic with respect to partial derivatives u_x, u_y, v_x, v_y, resulting that strain potential energy is quartic with respect to the partial derivatives. Thus, nodal force components based on Green strain are cubic. Let us derive cubic polynomials of nodal force components.

We introduce the following operator $\odot$ to describe quadratic and cubic terms in vector forms. Let $\boldsymbol{x} = [\, x_1, x_2, \cdots, x_m \,]^\top$ be an m-dimensional column vector and $\boldsymbol{y} = [\, y_1, y_2, \cdots, y_n \,]^\top$ be an n-dimensional column vector. Let $\boldsymbol{x} \odot \boldsymbol{y}$ be an mn-dimensional column vector defined as

$$\boldsymbol{x} \odot \boldsymbol{y} = \begin{bmatrix} x_1 \boldsymbol{y} \\ x_2 \boldsymbol{y} \\ \vdots \\ x_m \boldsymbol{y} \end{bmatrix} \tag{4.5.1}$$

This operator is associative; $\boldsymbol{x} \odot (\boldsymbol{y} \odot \boldsymbol{z}) = (\boldsymbol{x} \odot \boldsymbol{y}) \odot \boldsymbol{z}$, but not commutative. Additionally, we find

$$(\boldsymbol{a}^\top \boldsymbol{x})(\boldsymbol{b}^\top \boldsymbol{y}) = (\boldsymbol{a} \odot \boldsymbol{b})^\top (\boldsymbol{x} \odot \boldsymbol{y}) \tag{4.5.2}$$

$$(\boldsymbol{a}^\top \boldsymbol{x})(\boldsymbol{b}\boldsymbol{c}^\top \boldsymbol{z}) = \{\boldsymbol{b}\, (\boldsymbol{a} \odot \boldsymbol{c})^\top\} (\boldsymbol{x} \odot \boldsymbol{y}) \tag{4.5.3}$$

$$(\boldsymbol{a}^\top \boldsymbol{x})(\boldsymbol{y}^\top \boldsymbol{b}\boldsymbol{c}^\top \boldsymbol{z}) = (\boldsymbol{a} \odot \boldsymbol{b} \odot \boldsymbol{c})^\top (\boldsymbol{x} \odot \boldsymbol{y} \odot \boldsymbol{z}) \tag{4.5.4}$$

Note that $\boldsymbol{x} \odot \boldsymbol{y}$ consists of quadratic terms $x_i y_j$ and $\boldsymbol{x} \odot \boldsymbol{y} \odot \boldsymbol{z}$ consists of cubic terms $x_i y_j z_k$, suggesting that quadratic and cubic terms can be described in vector forms using operator $\odot$. Recall that

$$
\begin{aligned}
\frac{\partial U_p}{\partial \boldsymbol{\gamma}_u} &= P_u(\lambda I_\lambda + \mu I_\mu)\triangle h\, \boldsymbol{E} = (\lambda \triangle h) P_u I_\lambda \boldsymbol{E} + (\mu \triangle h) P_u I_\mu \boldsymbol{E} \\
\frac{\partial U_p}{\partial \boldsymbol{\gamma}_v} &= P_v(\lambda I_\lambda + \mu I_\mu)\triangle h\, \boldsymbol{E} = (\lambda \triangle h) P_v I_\lambda \boldsymbol{E} + (\mu \triangle h) P_v I_\mu \boldsymbol{E}
\end{aligned}
$$

where

$$
\begin{aligned}
P_u &= \begin{bmatrix} (1+u_x)\boldsymbol{a} & u_y \boldsymbol{b} & (1+u_x)\boldsymbol{b} + u_y \boldsymbol{a} \end{bmatrix} \\
P_v &= \begin{bmatrix} v_x \boldsymbol{a} & (1+v_y)\boldsymbol{b} & (1+v_y)\boldsymbol{a} + v_x \boldsymbol{b} \end{bmatrix}
\end{aligned}
$$

We describe $P_u I_\lambda \boldsymbol{E}$, $P_u I_\mu \boldsymbol{E}$, $P_v I_\lambda \boldsymbol{E}$, and $P_v I_\mu \boldsymbol{E}$ in cubic polynomials and combine the polynomials to obtain $\partial U_p / \partial \boldsymbol{\gamma}_u$ and $\partial U_p / \partial \boldsymbol{\gamma}_v$.

Let us calculate the first, second, and third order terms of $P_u I_\lambda \boldsymbol{E}$ with respect to $\boldsymbol{\gamma}_u$ and $\boldsymbol{\gamma}_v$. Note that

$$
\begin{aligned}
P_u I_\lambda \boldsymbol{E} &= \{(1+u_x)\boldsymbol{a} + u_y \boldsymbol{b}\}(E_{xx} + E_{yy}) \\
&= (\boldsymbol{a} + \boldsymbol{a} u_x + \boldsymbol{b} u_y) \left\{ u_x + v_y + \frac{1}{2}(u_x^2 + v_x^2 + u_y^2 + v_y^2) \right\}
\end{aligned}
$$

The first order terms are thus given by

$$
\boldsymbol{p}^1_{u\lambda} = \boldsymbol{a}(u_x + v_y) = \boldsymbol{a}\boldsymbol{a}^\top \boldsymbol{\gamma}_u + \boldsymbol{a}\boldsymbol{b}^\top \boldsymbol{\gamma}_v
$$

Letting $\boldsymbol{\gamma}_{uu} = \boldsymbol{\gamma}_u \odot \boldsymbol{\gamma}_u$, $\boldsymbol{\gamma}_{uv} = \boldsymbol{\gamma}_u \odot \boldsymbol{\gamma}_v$, $\boldsymbol{\gamma}_{vu} = \boldsymbol{\gamma}_v \odot \boldsymbol{\gamma}_u$, and $\boldsymbol{\gamma}_{vv} = \boldsymbol{\gamma}_v \odot \boldsymbol{\gamma}_v$, the second order terms are

$$
\begin{aligned}
\boldsymbol{p}^2_{u\lambda} &= (\boldsymbol{a} u_x + \boldsymbol{b} u_y)(u_x + v_y) + \frac{1}{2}\boldsymbol{a}(u_x^2 + v_x^2 + u_y^2 + v_y^2) \\
&= \frac{3}{2}\boldsymbol{a} u_x^2 + \frac{1}{2}\boldsymbol{a} u_y^2 + \boldsymbol{b} u_x u_y + \boldsymbol{a} u_x v_y + \boldsymbol{b} u_y v_y + \frac{1}{2}\boldsymbol{a} v_x^2 + \frac{1}{2}\boldsymbol{a} v_y^2 \\
&= \left\{ \frac{3}{2}\boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a})^\top + \frac{1}{2}\boldsymbol{a}(\boldsymbol{b} \odot \boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{a} \odot \boldsymbol{b})^\top \right\} \boldsymbol{\gamma}_{uu} \\
&\quad + \left\{ \boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{b})^\top \right\} \boldsymbol{\gamma}_{uv} + \left\{ \frac{1}{2}\boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a})^\top + \frac{1}{2}\boldsymbol{a}(\boldsymbol{b} \odot \boldsymbol{b})^\top \right\} \boldsymbol{\gamma}_{vv}
\end{aligned}
$$

Letting $\boldsymbol{\gamma}_{uuu} = \boldsymbol{\gamma}_u \odot \boldsymbol{\gamma}_u \odot \boldsymbol{\gamma}_u$ and $\boldsymbol{\gamma}_{uvv} = \boldsymbol{\gamma}_u \odot \boldsymbol{\gamma}_v \odot \boldsymbol{\gamma}_v$, the third order terms are

$$
\begin{aligned}
\boldsymbol{p}^3_{u\lambda} &= \frac{1}{2}(\boldsymbol{a} u_x + \boldsymbol{b} u_y)(u_x^2 + v_x^2 + u_y^2 + v_y^2) \\
&= \frac{1}{2}\boldsymbol{a}(u_x^3 + u_x v_x^2 + u_x u_y^2 + u_x v_y^2) + \frac{1}{2}\boldsymbol{b}(u_y u_x^2 + u_y v_x^2 + u_y^3 + u_y v_y^2) \\
&= \frac{1}{2}\left\{ \boldsymbol{a}(u_x^3 + u_x u_y^2) + \boldsymbol{b}(u_y u_x^2 + u_y^3) \right\} + \frac{1}{2}\left\{ \boldsymbol{a}(u_x v_x^2 + u_x v_y^2) + \boldsymbol{b}(u_y v_x^2 + u_y v_y^2) \right\} \\
&= \frac{1}{2}\left\{ \boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a} \odot \boldsymbol{a} + \boldsymbol{a} \odot \boldsymbol{b} \odot \boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{a} \odot \boldsymbol{a} + \boldsymbol{b} \odot \boldsymbol{b} \odot \boldsymbol{b})^\top \right\} \boldsymbol{\gamma}_{uuu} \\
&\quad + \frac{1}{2}\left\{ \boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a} \odot \boldsymbol{a} + \boldsymbol{a} \odot \boldsymbol{b} \odot \boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{a} \odot \boldsymbol{a} + \boldsymbol{b} \odot \boldsymbol{b} \odot \boldsymbol{b})^\top \right\} \boldsymbol{\gamma}_{uvv}
\end{aligned}
$$

Let us calculate the first, second, and third order terms of $P_u I_\mu \boldsymbol{E}$ with respect to $\boldsymbol{\gamma}_u$ and $\boldsymbol{\gamma}_v$. Note that

$$P_u I_\mu \boldsymbol{E} = (2\boldsymbol{a} + 2\boldsymbol{a}u_x)E_{xx} + 2\boldsymbol{b}u_y E_{yy} + (\boldsymbol{b} + \boldsymbol{b}u_x + \boldsymbol{a}u_y) \cdot 2E_{xy}$$

The first order terms are thus given by

$$\begin{aligned}
\boldsymbol{p}^1_{u\mu} &= 2\boldsymbol{a}u_x + \boldsymbol{b}(u_y + v_x) \\
&= (2\boldsymbol{a}\boldsymbol{a}^\top + \boldsymbol{b}\boldsymbol{b}^\top)\boldsymbol{\gamma}_u + \boldsymbol{b}\boldsymbol{a}^\top\boldsymbol{\gamma}_v
\end{aligned}$$

The second order terms are given by

$$\begin{aligned}
\boldsymbol{p}^2_{u\mu} &= \boldsymbol{a}(u_x^2 + v_x^2) + 2\boldsymbol{a}u_x^2 + 2\boldsymbol{b}u_y v_y + \boldsymbol{b}(u_x u_y + v_x v_y) + (\boldsymbol{b}u_x + \boldsymbol{a}u_y)(u_y + v_x) \\
&= 3\boldsymbol{a}u_x^2 + \boldsymbol{a}u_y^2 + 2\boldsymbol{b}u_x u_y + \boldsymbol{b}u_x v_x + \boldsymbol{a}u_y v_x + 2\boldsymbol{b}u_y v_y + \boldsymbol{a}v_x^2 + \boldsymbol{b}v_x v_y \\
&= \{3\boldsymbol{a}(\boldsymbol{a}\odot\boldsymbol{a})^\top + \boldsymbol{a}(\boldsymbol{b}\odot\boldsymbol{b})^\top + 2\boldsymbol{b}(\boldsymbol{a}\odot\boldsymbol{b})^\top\}\boldsymbol{\gamma}_{uu} \\
&+ \{\boldsymbol{b}(\boldsymbol{a}\odot\boldsymbol{a})^\top + \boldsymbol{a}(\boldsymbol{b}\odot\boldsymbol{a})^\top + 2\boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{b})^\top\}\boldsymbol{\gamma}_{uv} + \{\boldsymbol{a}(\boldsymbol{a}\odot\boldsymbol{a})^\top + \boldsymbol{b}(\boldsymbol{a}\odot\boldsymbol{b})^\top\}\boldsymbol{\gamma}_{vv}
\end{aligned}$$

The third order terms are given by

$$\begin{aligned}
\boldsymbol{p}^3_{u\mu} &= \boldsymbol{a}u_x(u_x^2 + v_x^2) + \boldsymbol{b}u_y(u_y^2 + v_y^2) + (\boldsymbol{b}u_x + \boldsymbol{a}u_y)(u_x u_y + v_x v_y) \\
&= \boldsymbol{a}u_x^3 + \boldsymbol{a}u_x v_x^2 + \boldsymbol{b}u_y^3 + \boldsymbol{b}u_y v_y^2 + \boldsymbol{b}u_x^2 u_y + \boldsymbol{b}u_x v_x v_y + \boldsymbol{a}u_y^2 u_x + \boldsymbol{a}u_y v_y v_x \\
&= \{\boldsymbol{a}(\boldsymbol{a}\odot\boldsymbol{a}\odot\boldsymbol{a} + \boldsymbol{b}\odot\boldsymbol{b}\odot\boldsymbol{a})^\top + \boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{b}\odot\boldsymbol{b} + \boldsymbol{a}\odot\boldsymbol{a}\odot\boldsymbol{b})^\top\}\boldsymbol{\gamma}_{uuu} \\
&+ \{\boldsymbol{a}(\boldsymbol{a}\odot\boldsymbol{a}\odot\boldsymbol{a} + \boldsymbol{b}\odot\boldsymbol{b}\odot\boldsymbol{a})^\top + \boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{b}\odot\boldsymbol{b} + \boldsymbol{a}\odot\boldsymbol{a}\odot\boldsymbol{b})^\top\}\boldsymbol{\gamma}_{uvv}
\end{aligned}$$

Let us calculate the first, second, and third order terms of $P_v I_\lambda \boldsymbol{E}$ with respect to $\boldsymbol{\gamma}_u$ and $\boldsymbol{\gamma}_v$. Note that

$$\begin{aligned}
P_v I_\lambda \boldsymbol{E} &= \{v_x\boldsymbol{a} + (1 + v_y)\boldsymbol{b}\}(E_{xx} + E_{yy}) \\
&= (\boldsymbol{b} + \boldsymbol{a}v_x + \boldsymbol{b}v_y)\left\{u_x + v_y + \frac{1}{2}(u_x^2 + v_x^2 + u_y^2 + v_y^2)\right\}
\end{aligned}$$

The first order terms are thus given by

$$\boldsymbol{p}^1_{v\lambda} = \boldsymbol{b}(u_x + v_y) = \boldsymbol{b}\boldsymbol{a}^\top\boldsymbol{\gamma}_u + \boldsymbol{b}\boldsymbol{b}^\top\boldsymbol{\gamma}_v$$

The second order terms are

$$\begin{aligned}
\boldsymbol{p}^2_{v\lambda} &= (\boldsymbol{a}v_x + \boldsymbol{b}v_y)(u_x + v_y) + \frac{1}{2}\boldsymbol{b}(u_x^2 + v_x^2 + u_y^2 + v_y^2) \\
&= \frac{1}{2}\boldsymbol{b}u_y^2 + \frac{1}{2}\boldsymbol{b}u_x^2 + \boldsymbol{b}u_x v_y + \boldsymbol{a}u_x v_x + \frac{3}{2}\boldsymbol{b}v_y^2 + \frac{1}{2}\boldsymbol{b}v_x^2 + \boldsymbol{a}v_y v_x \\
&= \left\{\frac{1}{2}\boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{b})^\top + \frac{1}{2}\boldsymbol{b}(\boldsymbol{a}\odot\boldsymbol{a})^\top\right\}\boldsymbol{\gamma}_{uu} + \left\{\boldsymbol{b}(\boldsymbol{a}\odot\boldsymbol{b})^\top + \boldsymbol{a}(\boldsymbol{a}\odot\boldsymbol{a})^\top\right\}\boldsymbol{\gamma}_{uv} \\
&+ \left\{\frac{3}{2}\boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{b})^\top + \frac{1}{2}\boldsymbol{b}(\boldsymbol{a}\odot\boldsymbol{a})^\top + \boldsymbol{a}(\boldsymbol{b}\odot\boldsymbol{a})^\top\right\}\boldsymbol{\gamma}_{vv}
\end{aligned}$$

Letting $\boldsymbol{\gamma}_{vuu} = \boldsymbol{\gamma}_v \odot \boldsymbol{\gamma}_u \odot \boldsymbol{\gamma}_u$ and $\boldsymbol{\gamma}_{vvv} = \boldsymbol{\gamma}_v \odot \boldsymbol{\gamma}_v \odot \boldsymbol{\gamma}_v$, the third order terms are

$$\begin{aligned}
\boldsymbol{p}^3_{v\lambda} &= \frac{1}{2}(\boldsymbol{a}v_x + \boldsymbol{b}v_y)(u_x^2 + v_x^2 + u_y^2 + v_y^2) \\
&= \frac{1}{2}\boldsymbol{a}(v_x u_x^2 + v_x^3 + v_x u_y^2 + v_x v_y^2) + \frac{1}{2}\boldsymbol{b}(v_y u_x^2 + v_y v_x^2 + v_y u_y^2 + v_y^3) \\
&= \frac{1}{2}\left\{\boldsymbol{a}(v_x u_x^2 + v_x u_y^2) + \boldsymbol{b}(v_y u_x^2 + v_y u_y^2)\right\} + \frac{1}{2}\left\{\boldsymbol{a}(v_x^3 + v_x v_y^2) + \boldsymbol{b}(v_y v_x^2 + v_y^3)\right\} \\
&= \frac{1}{2}\left\{\boldsymbol{a}(\boldsymbol{a}\odot\boldsymbol{a}\odot\boldsymbol{a} + \boldsymbol{a}\odot\boldsymbol{b}\odot\boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{a}\odot\boldsymbol{a} + \boldsymbol{b}\odot\boldsymbol{b}\odot\boldsymbol{b})^\top\right\}\boldsymbol{\gamma}_{vuu} \\
&+ \frac{1}{2}\left\{\boldsymbol{a}(\boldsymbol{a}\odot\boldsymbol{a}\odot\boldsymbol{a} + \boldsymbol{a}\odot\boldsymbol{b}\odot\boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{a}\odot\boldsymbol{a} + \boldsymbol{b}\odot\boldsymbol{b}\odot\boldsymbol{b})^\top\right\}\boldsymbol{\gamma}_{vvv}
\end{aligned}$$

Let us calculate the first, second, and third order terms of $P_v I_\mu \boldsymbol{E}$ with respect to $\boldsymbol{\gamma}_u$ and $\boldsymbol{\gamma}_v$. Note that

$$P_v I_\mu \boldsymbol{E} = 2\boldsymbol{a}v_x E_{xx} + (2\boldsymbol{b} + 2\boldsymbol{b}v_y)E_{yy} + (\boldsymbol{a} + \boldsymbol{b}v_x + \boldsymbol{a}v_y)\cdot 2E_{xy}$$

The first order terms are thus given by

$$\begin{aligned}
\boldsymbol{p}^1_{v\mu} &= 2\boldsymbol{b}v_y + \boldsymbol{a}(u_y + v_x) \\
&= \boldsymbol{a}\boldsymbol{b}^\top\boldsymbol{\gamma}_u + (2\boldsymbol{b}\boldsymbol{b}^\top + \boldsymbol{a}\boldsymbol{a}^\top)\boldsymbol{\gamma}_v
\end{aligned}$$

The second order terms are given by

$$\begin{aligned}
\boldsymbol{p}^2_{v\mu} &= 2\boldsymbol{a}u_x v_x + \boldsymbol{b}(u_y^2 + v_y^2) + 2\boldsymbol{b}v_y^2 + \boldsymbol{a}(u_x u_y + v_x v_y) + (\boldsymbol{b}v_x + \boldsymbol{a}v_y)(u_y + v_x) \\
&= \boldsymbol{b}u_y^2 + \boldsymbol{a}u_y u_x + \boldsymbol{a}u_y v_y + \boldsymbol{b}u_y v_x + 2\boldsymbol{a}u_x v_x + 3\boldsymbol{b}v_y^2 + \boldsymbol{b}v_x^2 + 2\boldsymbol{a}v_y v_x \\
&= \{\boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{b})^\top + \boldsymbol{a}(\boldsymbol{b}\odot\boldsymbol{a})^\top\}\boldsymbol{\gamma}_{uu} + \{\boldsymbol{a}(\boldsymbol{b}\odot\boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{a})^\top + 2\boldsymbol{a}(\boldsymbol{a}\odot\boldsymbol{a})^\top\}\boldsymbol{\gamma}_{uv} \\
&+ \{3\boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{a}\odot\boldsymbol{a})^\top + 2\boldsymbol{a}(\boldsymbol{b}\odot\boldsymbol{a})^\top\}\boldsymbol{\gamma}_{vv}
\end{aligned}$$

The third order terms are given by

$$\begin{aligned}
\boldsymbol{p}^3_{v\mu} &= \boldsymbol{a}v_x(u_x^2 + v_x^2) + \boldsymbol{b}v_y(u_y^2 + v_y^2) + (\boldsymbol{b}v_x + \boldsymbol{a}v_y)(u_x u_y + v_x v_y) \\
&= \boldsymbol{a}v_x u_x^2 + \boldsymbol{a}v_x^3 + \boldsymbol{b}v_y u_y^2 + \boldsymbol{b}v_y^3 + \boldsymbol{b}v_x u_x u_y + \boldsymbol{b}v_x^2 v_y + \boldsymbol{a}v_y u_y u_x + \boldsymbol{a}v_y^2 v_x \\
&= \{\boldsymbol{a}(\boldsymbol{a}\odot\boldsymbol{a}\odot\boldsymbol{a} + \boldsymbol{b}\odot\boldsymbol{b}\odot\boldsymbol{a})^\top + \boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{b}\odot\boldsymbol{b} + \boldsymbol{a}\odot\boldsymbol{a}\odot\boldsymbol{b})^\top\}\boldsymbol{\gamma}_{vuu} \\
&+ \{\boldsymbol{a}(\boldsymbol{a}\odot\boldsymbol{a}\odot\boldsymbol{a} + \boldsymbol{b}\odot\boldsymbol{b}\odot\boldsymbol{a})^\top + \boldsymbol{b}(\boldsymbol{b}\odot\boldsymbol{b}\odot\boldsymbol{b} + \boldsymbol{a}\odot\boldsymbol{a}\odot\boldsymbol{b})^\top\}\boldsymbol{\gamma}_{vvv}
\end{aligned}$$

Based on the above results, we can describe partial derivatives $\partial U_p/\partial\boldsymbol{\gamma}_u$ and $\partial U_p/\partial\boldsymbol{\gamma}_v$ in polynomial forms. Note that

$$\frac{\partial U_p}{\partial \boldsymbol{\gamma}_u} = (\lambda\triangle h)(\boldsymbol{p}^1_{u\lambda} + \boldsymbol{p}^2_{u\lambda} + \boldsymbol{p}^3_{u\lambda}) + (\mu\triangle h)(\boldsymbol{p}^1_{u\mu} + \boldsymbol{p}^2_{u\mu} + \boldsymbol{p}^3_{u\mu}) \tag{4.5.5}$$

$$\frac{\partial U_p}{\partial \boldsymbol{\gamma}_v} = (\lambda\triangle h)(\boldsymbol{p}^1_{v\lambda} + \boldsymbol{p}^2_{v\lambda} + \boldsymbol{p}^3_{v\lambda}) + (\mu\triangle h)(\boldsymbol{p}^1_{v\mu} + \boldsymbol{p}^2_{v\mu} + \boldsymbol{p}^3_{v\mu}) \tag{4.5.6}$$

Let us introduce 3×3 matrices A_1^p, A_2^p, B_1^p, and B_2^p defined by

$$\begin{bmatrix} A_1^p & A_2^p \\ B_1^p & B_2^p \end{bmatrix} = \lambda\triangle h \begin{bmatrix} \boldsymbol{a}\boldsymbol{a}^\top & \boldsymbol{a}\boldsymbol{b}^\top \\ \boldsymbol{b}\boldsymbol{a}^\top & \boldsymbol{b}\boldsymbol{b}^\top \end{bmatrix} + \mu\triangle h \begin{bmatrix} 2\boldsymbol{a}\boldsymbol{a}^\top + \boldsymbol{b}\boldsymbol{b}^\top & \boldsymbol{b}\boldsymbol{a}^\top \\ \boldsymbol{a}\boldsymbol{b}^\top & 2\boldsymbol{b}\boldsymbol{b}^\top + \boldsymbol{a}\boldsymbol{a}^\top \end{bmatrix} \tag{4.5.7}$$

Let us introduce 3×3^2 matrices $A^p_{1,1}$, $A^p_{1,2}$, $A^p_{2,2}$, $B^p_{1,1}$, $B^p_{1,2}$, and $B^p_{2,2}$ defined by

$$\begin{aligned} \begin{bmatrix} A^p_{1,1} & A^p_{1,2} & A^p_{2,2} \\ B^p_{1,1} & B^p_{1,2} & B^p_{2,2} \end{bmatrix} &= \lambda \triangle h \begin{bmatrix} C^{p,\lambda}_{2;\,1,1} & C^{p,\lambda}_{2;\,1,2} & C^{p,\lambda}_{2;\,1,3} \\ C^{p,\lambda}_{2;\,2,1} & C^{p,\lambda}_{2;\,2,2} & C^{p,\lambda}_{2;\,2,3} \end{bmatrix} \\ &\quad + \mu \triangle h \begin{bmatrix} C^{p,\mu}_{2;\,1,1} & C^{p,\mu}_{2;\,1,2} & C^{p,\mu}_{2;\,1,3} \\ C^{p,\mu}_{2;\,2,1} & C^{p,\mu}_{2;\,2,2} & C^{p,\mu}_{2;\,2,3} \end{bmatrix} \end{aligned} \tag{4.5.8}$$

where

$$\begin{aligned} C^{p,\lambda}_{2;\,1,1} &= \frac{3}{2}\boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a})^\top + \frac{1}{2}\boldsymbol{a}(\boldsymbol{b} \odot \boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{a} \odot \boldsymbol{b})^\top \\ C^{p,\lambda}_{2;\,1,2} &= \boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{b})^\top \\ C^{p,\lambda}_{2;\,1,3} &= \frac{1}{2}\boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a})^\top + \frac{1}{2}\boldsymbol{a}(\boldsymbol{b} \odot \boldsymbol{b})^\top \\ C^{p,\lambda}_{2;\,2,1} &= \frac{1}{2}\boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{b})^\top + \frac{1}{2}\boldsymbol{b}(\boldsymbol{a} \odot \boldsymbol{a})^\top \\ C^{p,\lambda}_{2;\,2,2} &= \boldsymbol{b}(\boldsymbol{a} \odot \boldsymbol{b})^\top + \boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a})^\top \\ C^{p,\lambda}_{2;\,2,3} &= \frac{3}{2}\boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{b})^\top + \frac{1}{2}\boldsymbol{b}(\boldsymbol{a} \odot \boldsymbol{a})^\top + \boldsymbol{a}(\boldsymbol{b} \odot \boldsymbol{a})^\top \end{aligned}$$

and

$$\begin{aligned} C^{p,\mu}_{2;\,1,1} &= 3\boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a})^\top + \boldsymbol{a}(\boldsymbol{b} \odot \boldsymbol{b})^\top + 2\boldsymbol{b}(\boldsymbol{a} \odot \boldsymbol{b})^\top \\ C^{p,\mu}_{2;\,1,2} &= \boldsymbol{b}(\boldsymbol{a} \odot \boldsymbol{a})^\top + \boldsymbol{a}(\boldsymbol{b} \odot \boldsymbol{a})^\top + 2\boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{b})^\top \\ C^{p,\mu}_{2;\,1,3} &= \boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a})^\top + \boldsymbol{b}(\boldsymbol{a} \odot \boldsymbol{b})^\top \\ C^{p,\mu}_{2;\,2,1} &= \boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{b})^\top + \boldsymbol{a}(\boldsymbol{b} \odot \boldsymbol{a})^\top \\ C^{p,\mu}_{2;\,2,2} &= \boldsymbol{a}(\boldsymbol{b} \odot \boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{a})^\top + 2\boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a})^\top \\ C^{p,\mu}_{2;\,2,3} &= 3\boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{a} \odot \boldsymbol{a})^\top + 2\boldsymbol{a}(\boldsymbol{b} \odot \boldsymbol{a})^\top \end{aligned}$$

Let us introduce 3×3^3 matrices C^p_1 and C^p_2 defined by

$$\begin{bmatrix} C^p_1 & C^p_2 \end{bmatrix} = \lambda \triangle h \begin{bmatrix} C^{p,\lambda}_3 & C^{p,\lambda}_3 \end{bmatrix} + \mu \triangle h \begin{bmatrix} C^{p,\mu}_3 & C^{p,\mu}_3 \end{bmatrix} \tag{4.5.9}$$

where

$$\begin{aligned} C^{p,\lambda}_3 &= \frac{1}{2}\left\{\boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a} \odot \boldsymbol{a} + \boldsymbol{a} \odot \boldsymbol{b} \odot \boldsymbol{b})^\top + \boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{a} \odot \boldsymbol{a} + \boldsymbol{b} \odot \boldsymbol{b} \odot \boldsymbol{b})^\top\right\} \\ C^{p,\mu}_3 &= \boldsymbol{a}(\boldsymbol{a} \odot \boldsymbol{a} \odot \boldsymbol{a} + \boldsymbol{b} \odot \boldsymbol{b} \odot \boldsymbol{a})^\top + \boldsymbol{b}(\boldsymbol{b} \odot \boldsymbol{b} \odot \boldsymbol{b} + \boldsymbol{a} \odot \boldsymbol{a} \odot \boldsymbol{b})^\top \end{aligned}$$

Partial derivatives $\partial U_p / \partial \boldsymbol{\gamma}_u$ and $\partial U_p / \partial \boldsymbol{\gamma}_v$ corresponding to triangle T_p are then described as follows:

$$\frac{\partial U_p}{\partial \boldsymbol{\gamma}_u} = A^p_1 \boldsymbol{\gamma}_u + A^p_2 \boldsymbol{\gamma}_v + A^p_{1,1} \boldsymbol{\gamma}_{uu} + A^p_{1,2} \boldsymbol{\gamma}_{uv} + A^p_{2,2} \boldsymbol{\gamma}_{vv} + C^p_1 \boldsymbol{\gamma}_{uuu} + C^p_2 \boldsymbol{\gamma}_{uvv} \tag{4.5.10}$$

$$\frac{\partial U_p}{\partial \boldsymbol{\gamma}_v} = B^p_1 \boldsymbol{\gamma}_u + B^p_2 \boldsymbol{\gamma}_v + B^p_{1,1} \boldsymbol{\gamma}_{uu} + B^p_{1,2} \boldsymbol{\gamma}_{uv} + B^p_{2,2} \boldsymbol{\gamma}_{vv} + C^p_1 \boldsymbol{\gamma}_{vuu} + C^p_2 \boldsymbol{\gamma}_{vvv} \tag{4.5.11}$$

Note that $\boldsymbol{\gamma}_u$ and $\boldsymbol{\gamma}_v$ are three-dimensional, $\boldsymbol{\gamma}_{uu}$, $\boldsymbol{\gamma}_{uv}$, and $\boldsymbol{\gamma}_{vv}$ consist of 3^2 components, and $\boldsymbol{\gamma}_{uuu}$, $\boldsymbol{\gamma}_{uvv}$, $\boldsymbol{\gamma}_{vuu}$, and $\boldsymbol{\gamma}_{vvv}$ have 3^3 components.

Let $\boldsymbol{c}_u = [\, u_1,\, u_2,\, \cdots,\, u_N \,]^\top$ and $\boldsymbol{c}_v = [\, v_1,\, v_2,\, \cdots,\, v_N \,]^\top$ be collective vectors of displacement components. Collective vectors corresponding to quadratic terms are $\boldsymbol{c}_{uu} = \boldsymbol{c}_u \odot \boldsymbol{c}_u$, $\boldsymbol{c}_{uv} = \boldsymbol{c}_u \odot \boldsymbol{c}_v$, $\boldsymbol{c}_{vu} = \boldsymbol{c}_v \odot \boldsymbol{c}_u$, and $\boldsymbol{c}_{vv} = \boldsymbol{c}_v \odot \boldsymbol{c}_v$. Collective vectors corresponding to cubic terms are $\boldsymbol{c}_{uuu} = \boldsymbol{c}_u \odot \boldsymbol{c}_u \odot \boldsymbol{c}_u$, $\boldsymbol{c}_{uvv} = \boldsymbol{c}_u \odot \boldsymbol{c}_v \odot \boldsymbol{c}_v$, $\boldsymbol{c}_{vuu} = \boldsymbol{c}_v \odot \boldsymbol{c}_u \odot \boldsymbol{c}_u$, and $\boldsymbol{c}_{vvv} = \boldsymbol{c}_v \odot \boldsymbol{c}_v \odot \boldsymbol{c}_v$. Synthesizing above equations over all triangles yield partial derivatives $\partial U / \partial \boldsymbol{c}_u$ and $\partial U / \partial \boldsymbol{c}_v$ described as

$$\frac{\partial U}{\partial \boldsymbol{c}_u} = A_1 \boldsymbol{c}_u + A_2 \boldsymbol{c}_v + A_{1,1} \boldsymbol{c}_{uu} + A_{1,2} \boldsymbol{c}_{uv} + A_{2,2} \boldsymbol{c}_{vv} + C_1 \boldsymbol{c}_{uuu} + C_2 \boldsymbol{c}_{uvv} \tag{4.5.12}$$

$$\frac{\partial U}{\partial \boldsymbol{c}_v} = B_1 \boldsymbol{c}_u + B_2 \boldsymbol{c}_v + B_{1,1} \boldsymbol{c}_{uu} + B_{1,2} \boldsymbol{c}_{uv} + B_{2,2} \boldsymbol{c}_{vv} + C_1 \boldsymbol{c}_{vuu} + C_2 \boldsymbol{c}_{vvv} \tag{4.5.13}$$

where

$$A_i = \bigoplus_p A_i^p, \qquad B_i = \bigoplus_p B_i^p, \qquad C_i = \bigoplus_p C_i^p, \quad (i = 1,\, 2) \tag{4.5.14}$$

$$A_{i,j} = \bigoplus_p A_{i,j}^p, \quad B_{i,j} = \bigoplus_p B_{i,j}^p, \quad (i,j) = (1,1), (1,2), (2,2) \tag{4.5.15}$$

Operator $\oplus$ calculates total coefficient matrices by synthesizing coefficient matrices at individual triangles.

Note that coefficient matrices A_i, B_i, $A_{i,j}$, $B_{i,j}$, and C_i can be computed in advance. Then, we can calculate Green strain based nodal force vector by eqs. (4.5.12)(4.5.13), which consist of simple vector/matrix operations.

Implementation One barrier to implement the above calculation is memory consumption. Let n_p be the number of nodal points. Collective vectors for quadratic terms $\boldsymbol{c}_{uu}$, $\boldsymbol{c}_{uv}$, and $\boldsymbol{c}_{vv}$ are n_p^2-dimensional and collective vectors for cubic terms $\boldsymbol{c}_{uuu}$, $\boldsymbol{c}_{uvv}$, $\boldsymbol{c}_{vuu}$, and $\boldsymbol{c}_{vvv}$ are n_p^3-dimensional. Then, we find that A_i and B_i are $n_p \times n_p$ matrices, $A_{i,j}$ and $B_{i,j}$ are $n_p \times n_p^2$ matrices, and C_1 and C_2 are $n_p \times n_p^3$ matrices, which consume excessive memory. Here we reduce dimensions of collective vectors to prevent excessive memory consumption.

Recall that total strain potential energy is the sum of strain potential energies at individual triangles, implying that quadratic and cubic terms originate from triangles. This implies that quadratic term $u_i u_j$ appears if there exists a triangle that involves both P_i and P_j, and that cubic term $u_i u_j u_k$ appears if there exists a triangle that involves all P_i, P_j, and P_k. In other words, quadratic term $u_i u_j$ does not appear if no triangles involve both P_i and P_j, and cubic term $u_i u_j u_k$ does not appear if no triangles involve all P_i, P_j, and P_k. We can eliminate such non-existing terms and their corresponding columns of coefficient matrices. For example, when no triangles involve both P_i and P_j, column of matrix $A_{1,1}$ corresponding to $u_i u_j$ can be eliminated. Similarly, when no triangles involve all P_i, P_j, and P_k, column of matrix C_1 corresponding to $u_i u_j u_k$ can be eliminated. Such elimination yields compact collective vectors and compact coefficient matrices.

Let us take a simple example of rectangle region $\square \mathrm{P}_1 \mathrm{P}_3 \mathrm{P}_6 \mathrm{P}_4$ shown in Fig. 2.2. In this example, vectors $\boldsymbol{c}_u$, $\boldsymbol{c}_{uu}$, and $\boldsymbol{c}_{uuu}$ are described as follows:

$$\boldsymbol{c}_u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}, \qquad \boldsymbol{c}_{uu} = \begin{bmatrix} u_1u_1 \\ u_1u_2 \\ u_1u_4 \\ u_2u_1 \\ u_2u_2 \\ u_2u_3 \\ u_2u_4 \\ u_2u_5 \\ \vdots \\ u_6u_6 \end{bmatrix}, \qquad \boldsymbol{c}_{uuu} = \begin{bmatrix} u_1u_1u_1 \\ u_1u_1u_2 \\ u_1u_1u_4 \\ u_1u_2u_1 \\ u_1u_2u_2 \\ u_1u_2u_4 \\ u_1u_4u_1 \\ u_1u_4u_2 \\ u_1u_4u_4 \\ \vdots \\ u_6u_6u_6 \end{bmatrix}$$

Note that suffices appear in lexicographic order. Quadratic term u_1u_3 is not involved in $\boldsymbol{c}_{uu}$ as no triangles include both P_1 and P_3. Cubic term $u_1u_2u_5$ is not involved in $\boldsymbol{c}_{uuu}$ as no triangles include all P_1, P_2, and P_5. Dimension of vector $\boldsymbol{c}_{uu}$ is 24, which is less than 6^2, and dimension of vector $\boldsymbol{c}_{uuu}$ is 84, which is less than 6^3. Let n_e, and n_t be the numbers of edges and triangles. Noting that $n_p = 6$, $n_e = 9$, and $n_t = 4$ in this example, we find that $n_p + 2n_e = 24$ and $n_p + 6n_e + 6n_t = 84$ (see Problem 3). Other collective vectors can be described in these compact forms. Then, coefficient matrices can be described in compact forms: A_i and B_i are 6×6 matrices, $A_{i,j}$ and $B_{i,j}$ are 6×24 matrices, and C_1 and C_2 are 6×84 matrices.

Let `c2i` and `c2j` be vectors consisting of the first and second suffices of quadratic terms in $\boldsymbol{c}_{uu}$: `c2i` $= [1, 1, 1, 2, 2, 2, 2, 2, \cdots, 6]^\top$ and `c2j` $= [1, 2, 4, 1, 2, 3, 4, 5, \cdots, 6]^\top$. Then, collective vectors for quadratic terms are calculated by $\boldsymbol{c}_{uu} = \boldsymbol{c}_u$(`c2i`) `.*` $\boldsymbol{c}_u$(`c2j`), $\boldsymbol{c}_{uv} = \boldsymbol{c}_u$(`c2i`) `.*` $\boldsymbol{c}_v$(`c2j`), and $\boldsymbol{c}_{vv} = \boldsymbol{c}_v$(`c2i`) `.*` $\boldsymbol{c}_v$(`c2j`). Letting `c3i`, `c3j`, and `c3k` be vectors consisting of the first, second, and third suffices of cubic terms in $\boldsymbol{c}_{uuu}$: `c3i` $= [1, 1, 1, 1, 1, 1, 1, 1, 1, \cdots, 6]^\top$, `c3j` $= [1, 1, 1, 2, 2, 2, 4, 4, 4, \cdots, 6]^\top$, and `c3k` $= [1, 2, 4, 1, 2, 4, 1, 2, 4, \cdots, 6]^\top$, collective vectors for cubic terms are calculated similarly such as $\boldsymbol{c}_{uuu} = \boldsymbol{c}_u$(`c3i`) `.*` $\boldsymbol{c}_u$(`c3j`) `.*` $\boldsymbol{c}_u$(`c3k`) and $\boldsymbol{c}_{uvv} = \boldsymbol{c}_u$(`c3i`) `.*` $\boldsymbol{c}_v$(`c3j`) `.*` $\boldsymbol{c}_v$(`c3k`).

Let us describe pairs and triplets of suffices by $(n_p + 1)$-base numbers. In this example, a pair of suffices i and j is described by a 7-base number $7i + j$. Vector `id2` consists of these numbers. Finding number $7i + j$ in vector `id2` yields the row number in collective vector $\boldsymbol{c}_{uu}$ corresponding to quadratic term u_iu_j. A triplet of suffices i, j, and k is described by a 7-base number $7^2i + 7j + k$. Vector `id3` consists of these numbers. Finding number $7^2i + 7j + k$ in vector `id3` yields the row number in collective vector $\boldsymbol{c}_{uuu}$ corresponding to cubic term $u_iu_ju_k$.

Let us demonstrate how operator $\oplus$ in eqs. (4.5.14)(4.5.15) works in this example. Recall that matrix $C^{1,\,\lambda}_{2;\,1,1}$ originates from triangle T_1, which consists of three nodal points P_1, P_2, and P_4. We find matrix $C^{1,\,\lambda}_{2;\,1,1}$ is a 3×9 matrix given by

$$C^{1,\,\lambda}_{2;\,1,1} = \frac{1}{2000} \begin{bmatrix} -6 & 3 & 3 & 5 & -3 & -2 & 1 & 0 & -1 \\ 4 & -3 & -1 & -3 & 3 & 0 & -1 & 0 & 1 \\ 2 & 0 & -2 & -2 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}$$

Columns are related to quadratic terms u_1u_1, u_1u_2, u_1u_4, u_2u_1, u_2u_2, u_2u_4, u_4u_1, u_4u_2, and u_4u_4, which correspond to the 1, 2, 3, 4, 5, 7, 13, 14, and 15-th elements of collective vector c_{uu}. So, the column vectors of $C^{1,\,\lambda}_{2;\,1,1}$ contribute to the 1, 2, 3, 4, 5, 7, 13, 14, and 15-th columns of the total coefficient matrix. Column numbers in the total coefficient matrix can be obtained by finding the 7-base number corresponding to quadratic terms in vector `id2`. Matrix $C_3^{1,\,\lambda}$, which corresponds to triangle T_1, is a 3×27 matrix. The first, second, and third columns are corresponding to $u_1 u_1 u_1$, $u_1 u_1 u_2$, and $u_1 u_1 u_4$. So, these column vectors of $C_3^{1,\,\lambda}$ contribute to the first, second, and third columns of the total coefficient matrix. Column numbers in the total coefficient matrix can be obtained by finding the 7-base number corresponding to cubic terms in vector `id3`. Method **calculate_coefficient_matrices_for_Green_strain** of class `Body` prepares coefficient matrices A_i, B_i, $A_{i,j}$, $B_{i,j}$, and C_i in compact forms and vectors `c2i`, `c2j`, `c3i`, `c3j`, and `c3k` for calculating compact collective vectors.

Problems

1. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be independent vectors, $d = \|\boldsymbol{y} - \boldsymbol{x}\|$, and

$$\boldsymbol{e} = \frac{\boldsymbol{y} - \boldsymbol{x}}{d}$$

   Show $\partial d / \partial \boldsymbol{y} = \boldsymbol{e}$ and $\partial d / \partial \boldsymbol{x} = -\boldsymbol{e}$.

2. Simulate the dynamic behavior of an elastic ring rolling on a flat floor based on Cauchy strain. Compare the result with computed result based on Green strain.

3. Explain why dimension of compact collective vector $\boldsymbol{c}_{uu}$ is equal to $n_p + 2n_e$. Explain why dimension of compact collective vector $\boldsymbol{c}_{uuu}$ is equal to $n_p + 6n_e + 6n_t$.

4. Observe how tangential damping coefficient B_{t} affects dynamic behavior of an elastic ring rolling on a flat floor.