Modeling of Deformable Strings with Bend, Twist, and Extension in 3D Space

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Abstract

A systematic approach to the modeling of deformable strings such as cords and ropes is presented. There exist many manipulative operations that deal with deformable objects in the environments that robots are expected to take active parts, while rigid object manipulation has been a main interest in most researches on robotic manipulation. Manipulative operations that deal with deformable objects is thus a challenging issue in robotic manipulation.

In this article, we will present a static formulation of the deformation of strings. First, a generalized coordinate system appropriate to describe the string deformation is introduced. Secondly, internal energy of a string and geometric constraints imposed on it are formulated. Deformation of the string is then computed by use of nonlinear programming techniques. Finally, numerical examples and experimental results demonstrate the effectiveness of the proposed approach.

1 Introduction

In the past decades, many researchers have been interested in robotic manipulation and many approaches and methods have been presented. Most of these works focus on manipulation of rigid objects. Namely, manipulative operations such as grasping, pick-and-place operation, assembly, and disassembly of rigid objects have been taken into consideration. Investigating manufacturing fields and viewing our living environment, however, there are many operations that deal with deformable soft objects. For example, many manufacturing processes deal with deformable objects such as rubber tubes, sheet metals, cords, leather products, and paper sheets. There exist many deformable soft objects such as clothes and foods in our daily life. Soft tissues including muscles and skin are manipulated in medical operations. Robotic machine systems are expected to take active parts in these environments. Manipulation of deformable object manipulation is thus an important research issue.

Automatic handling of deformable parts in shoe and garment manufacturing have been studied [1]. These studies have been done for individual processes independently and few systematic approaches have been developed yet. Solid mechanics has been studied for a long time in order to analyze deformation of a solid body by investigating the relationship between stress and strain of the object [2]. It is not easy to analyze large deformation of a soft object such as paper and leather by solid mechanics approach, which basically deals with small deformation of a solid body. In computer graphics, some methods to represent shapes of curved lines and curved surfaces has been proposed [3]. Shape of cloths [4] and shape of elastic objects [5] have also been studied. These studies are not applicable to manipulative operations of deformable objects directly since they mainly focus on deformed shapes of objects and manipulation processes are not formulated there.

Solid modeling techniques [6] have been applied to the studies on manipulation of rigid objects so that the model of the manipulated objects can be built. Thanks to the solid modeling techniques, a systematic approach to the manipulation of rigid objects has been developed recently. On the contrary, we have no systematic method of modeling deformable objects during their manipulative operations.

In this article, a systematic approach to the formulation of deformable strings such as cords and ropes is presented. First, a generalized coordinate system is introduced to express the deformation of a string. Secondly, internal energy of a string and geometric constraints imposed on it are formulated. Deformation of the string can be computed by minimizing the internal energy under the geometric constraints. Next, an algorithm to compute the deformation is developed based on nonlinear programming techniques. Finally, simple experimental results are shown to demonstrate how the deformation is computed by use of the proposed approach.

2 Formulation of String Deformation

2.1 Representation of Deformation

In this section, we will formulate the deformation of a string in three-dimensional space. The deformation is formulated by the following steps:

Step 1. Introduce generalized coordinates that can describe the natural shape and the deformed shape of the string.

Step 2. Formulate physical quantities of the string.

Step 3. Formulate interactions with other objects surrounding the string.

Let us introduce the generalized coordinate system expressing the deformation. Let L be the length of the object and s be the distance from one endpoint of the object along it. In order to describe the object shape, we will introduce a coordinate system fixed on space; $O - xyz$. Let $\boldsymbol{x}(s) = [x(s), y(s), z(s)]^T$ be spatial coordinates corresponding to a point $P(s)$ on the object. Now, let us focus on the bend deformation of the object by ignoring its extensional deformation. Then, the magnitude of the derivative of $\boldsymbol{x}(s)$ with respect to s must be equal to 1, that is, $\|d\boldsymbol{x}/ds\| = 1$, since the object has no extensional deformation. In order to describe the bend deformation of a string, we will introduce a local object coordinates, say $P - \xi\eta\zeta$, at individual points on the string, as shown in Figure 1. Select the direction of coordinates so that the ξ-axis, η-axis, and ζ-axis are parallel to x-axis, y-axis, and z-axis, respectively, in natural state. Bend deformation of the string is then given by the relationship between the local coordinates at each point and the global coordinates. Let us describe the orientation of the local coordinate system with respect to the space coordinate system by use of Eulerian angles, $\phi(s)$, $\theta(s)$, and $\psi(s)$. The rotational transformation from $P - \xi\eta\zeta$ to $O - xyz$ is expressed by the following rotational matrix:

$$\begin{bmatrix} \mathrm{C}_\theta \mathrm{C}_\phi \mathrm{C}_\psi - \mathrm{S}_\phi \mathrm{S}_\psi & \mathrm{C}_\theta \mathrm{S}_\phi \mathrm{C}_\psi + \mathrm{C}_\phi \mathrm{S}_\psi & -\mathrm{S}_\theta \mathrm{C}_\psi \\ -\mathrm{C}_\theta \mathrm{C}_\phi \mathrm{S}_\psi - \mathrm{S}_\phi \mathrm{C}_\psi & -\mathrm{C}_\theta \mathrm{S}_\phi \mathrm{S}_\psi + \mathrm{C}_\phi \mathrm{C}_\psi & \mathrm{S}_\theta \mathrm{S}_\psi \\ \mathrm{S}_\theta \mathrm{C}_\phi & \mathrm{S}_\theta \mathrm{S}_\phi & \mathrm{C}_\theta \end{bmatrix}$$

For the sake of simplicity, $\cos\theta$ and $\sin\theta$ are abbreviated as C_θ and S_θ, respectively. A unit vector along ζ-axis at the natural state are transformed into the following vector due to the object deformation:

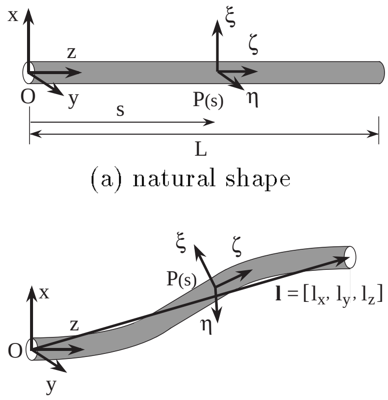
$$\boldsymbol{\zeta}(s) \triangleq \begin{bmatrix} -\sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}. \tag{1}$$

Since the above vector coincides to the derivative $d\boldsymbol{x}/ds$, the spatial coordinates can be computed by integrating it. Namely,

$$\boldsymbol{x}(s) = \int_0^s \boldsymbol{\zeta}(s) ds + \boldsymbol{x}_0 \tag{2}$$

where $\boldsymbol{x}_0 = [x_0, y_0, z_0]^T$ denotes the spatial coordinates at the endpoint corresponding to $s = 0$. Note that this representation satisfies $\|d\boldsymbol{x}/ds\| = 1$.

Extensional deformations can be taken into consideration by introducing a strain at each point $P(s)$. Let ε be extensional strain at point $P(s)$ on a string along its central axis. A unit vector along ζ-axis at the natural state are transformed into $(1 - \varepsilon)\boldsymbol{\zeta}(s)$ due to the object deformations. The spatial coordinates are computed by integrating $(1-\varepsilon)\boldsymbol{\zeta}(s)$ instead of $\boldsymbol{\zeta}(s)$ in eq.(2).



(a) natural shape

(b) deformed shape

Figure 1: Description of relationship between natural shape and deformed shape

From the above discussion, we find that the geometrical shape of a deformable string can be represented by four variables, that is, Eulerian angles ϕ, θ, and ψ as well as extensional strain ε. Note that each variable depends upon parameter s.

2.2 Internal Energy of String

Variational principles developed in analytical mechanics are useful to formulate physical properties of a deformable string using the introduced generalized coordinates [7]. In this article, we will derive a statically stable shape of a string by applying the variational principle for statics. Dynamical effects during operations is assumed to be negligible. Let U be the potential energy of a string and W be the work done by external forces applied to the string. The variational principle for statics is given by

$$\delta(U - W) = 0 \tag{3}$$

where δ denotes variational operator. The above equation implies that the internal energy $U - W$ of the string reaches to its minimum at its statically stable shape. In other words, the stable shape can be computed by solving the minimization problem.

Let us first formulate the potential energy of a string. Assume that the thickness and the width of the string is negligibly small. Applying Bernoulli and Navier's assumption, the potential energy U is described as follows:

$$U = U_{flex} + U_{tor} + U_{ext} + U_{grav} \tag{4}$$

where U_{flex}, U_{tor}, and U_{ext} represent flexural energy, torsional energy, and extensional energy of the string, respectively, and U_{grav} denotes its gravitational energy.

Let us describe the curvature of a string and its torsional angle, which are originated from differential geometry [8], in order to express bend and twist deformations. Let κ and ω be the curvature and the torsional angle at point $P(s)$, respectively. The curvature and the torsional angle can be described by use of Eulerian angles as follows:

$$\kappa^2 = \left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2,$$
$$\omega^2 = \left(\frac{d\phi}{ds}\cos\theta + \frac{d\psi}{ds}\right)^2.$$

Assume that bending moment and twisting moment are proportional to curvature and torsional angle at each point $P(s)$, respectively, over the object. The flexural energy and the torsional energy are then described as follows:

$$U_{flex} = \int_0^L \frac{1}{2} R_f \kappa^2 ds,$$
$$U_{tor} = \int_0^L \frac{1}{2} R_t \omega^2 ds$$

where R_f and R_t represent the flexural rigidity and the torsional rigidity at point $P(s)$, respectively. Assuming that extensional force is proportional to external strain at each point $P(s)$, extensional energy is given as follows:

$$U_{ext} = \int_0^L \frac{1}{2} R_e \varepsilon^2 ds$$

where R_e denotes the extensional rigidity of the object. The gravitational energy is given by

$$U_{grav} = \int_0^L Dx\, ds$$

where D represents weight per unit length of the object. Note that quantities R_f, R_t, R_e, and D may vary with respect to variable s.

Finally, let us formulate the work done by external forces. Suppose that an external force $\boldsymbol{F}_k$ is applied to a string at point $P(s_k)$. Note that coordinates corresponding to $P(s_k)$ at natural shape are given by $\boldsymbol{x}_0(s_k) = [0, 0, s_k]^T$. Thus, the work done by force $\boldsymbol{F}_k$ is described as $\boldsymbol{F}_k^T\{\boldsymbol{x}(s_k) - \boldsymbol{x}_0(s_k)\}$. Assuming that n external forces are applied to the object, the resultant work done by these forces is described as follows:

$$W = \sum_{k=1}^{n} \boldsymbol{F}_k^T \{\boldsymbol{x}(s_k) - \boldsymbol{x}_0(s_k)\} \tag{5}$$

where $\boldsymbol{F}_1$ through $\boldsymbol{F}_n$ are predefined forces acting on the object at point $P(s_1)$ through $P(s_n)$, respectively.

2.3 Geometrical Constraints

Due to the interaction between a string and other objects such as fingertips and obstacles, some geometric constraints are imposed on the string. Let us derive the geometric constraints imposed on a string. The relative position between some points on a string is often controlled during manipulative operations. Consider a constraint that specifies the positional relationship between two points on the string. Let $\boldsymbol{l} = [l_x, l_y, l_z]^T$ be a predetermined vector describing the relative position between two operational points, $P(s_a)$ and $P(s_b)$. The following equational condition must be then satisfied:

$$\boldsymbol{x}(s_b) - \boldsymbol{x}(s_a) = \boldsymbol{l}. \tag{6}$$

The orientation at some points of the string must be also controlled during the operation. These orientational constraints are simply described as follows:

$$\phi(s_c) = \phi_c, \quad \theta(s_c) = \theta_c, \quad \psi(s_c) = \psi_c \tag{7}$$

where ϕ_c, θ_c, and ψ_c are predefined angles at one operational point $P(s_c)$.

Contact between a string and rigid obstacles in operation space also yields other geometric constraints. Note that any points on the string must be located outside each obstacle or on it. Let us describe the surface of an obstacle fixed on space by equation $h(\boldsymbol{x}) = 0$. Assume that value of function $h(\boldsymbol{x})$ is positive inside the obstacle and is negative outside it. The condition that a string is not interfered with this obstacle is then described as follows:

$$h(\boldsymbol{x}(s)) \le 0, \quad \forall s \in [0, L]. \tag{8}$$

Note that the condition that a string is not interfered with obstacles is described by a set of inequalities, since mechanical contacts between the objects constrain the object motion unidirectionally.

From the above discussion, we find that the geometric constraints imposed on a string are given by not only equational conditions such as eqs.(6) and (7) but also inequality conditions such as eq.(8). The deformed shape of the object is, therefore, determined by minimizing internal energy $U - W$ under these geometric constraints. Namely, computation of object deformation results in a variational problem under equational and inequality conditions.

3 Computation Algorithm

Computation of the deformation of a string results in a variational problem, as mentioned in the previous section. One method to solve a variational problem is Euler's approach, which is based on the stationary condition in function space. Recall that the geometric constraints resulting from mechanical contacts are unidirectional and are mathematically described by inequalities such as eq.(8). These conditions are nonholonomic constraints [9]. Thus, the shape of a string that minimizes potential energy does not necessarily satisfy the stationary condition. This implies that Euler's approach, which is based on the stationary condition, is not applicable.

In this paper, we will develop a direct method based on Ritz's method [10] and a nonlinear programming technique. Let us express functions $\phi(s)$, $\theta(s)$, $\psi(s)$, and $\varepsilon(s)$ by linear combinations of basic functions $\varphi_1(s)$ through $\varphi_n(s)$. Substituting the linear combinations, internal energy $U - W$ is described by a function of the coefficients of the linear combinations. The geometric constraints are also described by conditions involving the coefficients. In addition, discretizing eq.(8) by dividing interval $[0, L]$ into N small intervals yields a finite number of conditions. As a result, a set of the geometric constraints is expressed by equations and inequalities with respect to the coefficients. The deformation of a string can be then derived by computing the coefficients that minimize the internal energy under the geometric constraints. This minimization problem under equality and inequality conditions can be solved by use of a nonlinear programming technique such as multiplier method [11].

4 Numerical Examples

In this section, some numerical examples are shown in order to demonstrate how the proposed method computes the deformation of a string. The following set of basic functions are used in the computation of these examples:

$$\varphi_1 = 1, \quad \varphi_2 = s,$$
$$\varphi_{2n+1} = \sin\frac{2n\pi s}{L},$$
$$\varphi_{2n+2} = \cos\frac{2n\pi s}{L}. \quad (n = 1, 2, 3, 4)$$

In the nonlinear optimization for the computation of deformed shapes, multiplier method and Nelder and Mead's simplex method are applied.

Transition among Shapes The first example shows the deformation of a string computed by considering its bending and torsion, say, $U = U_{flex} + U_{tor}$. Let us reduce a string of its length L along the central axis of the object. Suppose that the orientation at one endpoint $P(0)$ is fixed while the rotation around the central axis of the object alone is allowed at the other endpoint $P(L)$. Then, we have the following constraints:

$$\phi(0) = \theta(0) = \psi(0) = 0,$$
$$\sin\theta(L) = 0, \quad \cos\theta(L) = 1.$$

Assume that dimensionless quantity R_f/R_t, which characterizes the string shape, is equal to 100. Let us show the computed shapes corresponding to various values of the distance between two endpoints; $0.8L$, $0.7L$, $0.6L$, $0.5L$, $0.4L$, and $0.3L$. Computed shapes of the string are shown in Figure 2. Since the string shape is not planar for some values of the distance, the top view, the front view, and the side view are shown in the figure. The shape of the string is involved in x-z plane when the distance is equal to $0.8L$ or $0.7L$. The string is twisted and is not involved in any plane when the distance is equal to $0.6L$ or $0.5L$. The string contains one knot when the distance is equal to $0.4L$ or $0.3L$. Thus, it turns out that the string shape transits from a knot-free shape into a one-knot shape as the distance between the endpoints decreases. Recall that the direction along the central axis of the string is fixed at both endpoints. This implies that the string must have a non-planar shape during this transition.

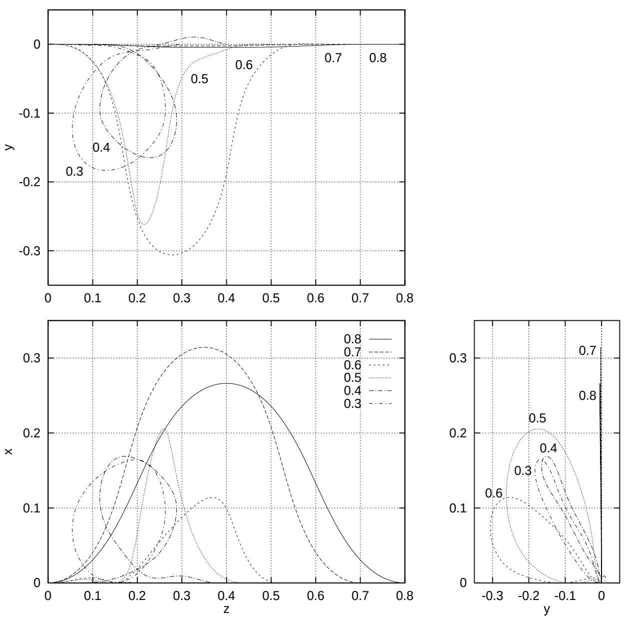


Figure 2: Example of computed deformation with shape transitions

Effect of Gravity The second example demonstrates the computation of string shapes considering gravitational energy. Assume that the potential energy of a string consists of flexural energy and gravitational energy, say, $U = U_{flex} + U_{grav}$. Normalizing the potential energy and the geometric constraints by means of dividing variable s by length L, we find that the shape of the object is determined by the following dimensionless quantity:

$$\rho = \frac{D}{R_f}L^3$$

Quantity ρ represents the contribution of the gravitational force to the shape of a string. Especially, the gravitational force is neglected at $\rho = 0.0$. The distance between the endpoints l is fixed to 70 and both angles from the horizon at these points θ_0 and θ_L are equal to $0(rad)$.

The deformed shapes of a string corresponding to various values of ρ; 0.0, 1.0, 2.0, 3.0, 5.0 $(\times 10^3)$ are shown in Figure 3. As shown in the figure, the height of the string decreases with increasing quantity ρ. In addition, the deformed shape is not symmetric any more when ρ exceeds 2.0×10^3. Note that we have two shapes symmetric each other with respect to the central vertical line in these cases. One shape of the two is illustrated in the figure. In order to verify that the unsymmetrical shape minimizes the potential energy, let us compute the potential energy of the string assuming that the string shape is symmetric. We find that potential energy U is, for example, equal to 0.746 at $\rho = 3.0 \times 10^3$ assuming that the deformed shape is symmetric, while the minimum value of potential en-

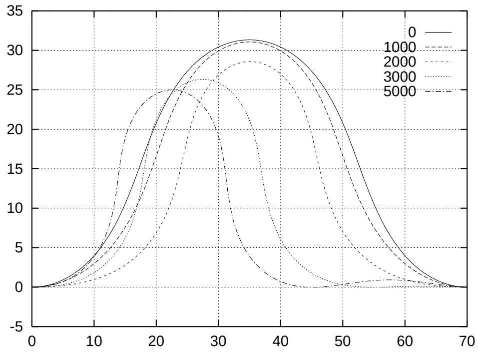


Figure 3: Example of computed object shapes considering gravity

ergy is equal to 0.518. Namely, the symmetric shape does not satisfy the condition that the potential energy reaches its minimum at a stable deformed shape. This implies that deformed shapes are unsymmetrical when dimensionless quantity ρ exceeds a certain value.

Applying External Force and Moment The third example shows the deformation caused by external force and external moment. Let us assume that a string deforms along z-x plane and its potential energy U consists of flexural energy U_{flex} alone. Initial shape of a string and geometric constraints imposed on the string are illustrated in Figure 4. Assume that no moment can be exerted at both endpoints. The right endpoint may lose contact with the right fingertip. The constraints are then described as follows:

$$\begin{aligned} &\dot{\theta}(0) = \dot{\theta}(L) = 0, \\ &x(L) = 0, \quad z(L) - l \le 0, \\ &x(s) \ge 0, \quad \forall s \in [0, L] \end{aligned}$$

Reduce a string of its length L to $l = 0.8L$ beforehand. Then, let us exert an external force at a point corresponding to $s_{\text{ex}} = 0.5L$ in the direction of $\theta_n = 3\pi/4$ from the horizon. The deformed shapes are plotted in Figure 5-(a). From this figure, we find that the contact at the right endpoint is lost when the magnitude of the force exceeds $0.14R_f/L$. Next, let us exert an external moment to the initial shape where $l = 0.8L$ around the y-axis at a point corresponding to $s_{\text{ex}} = 0.5L$. The deformed shapes are plotted in Figure 5-(b). From this figure, we find that the contact at the right endpoint is lost when the magnitude of the moment exceeds $0.17R_f$.

5 Experimental Results

In this section, we will compare the measured deformation and the computed deformation in order to demonstrate the validity of the proposed method. Note that the proposed method can be applied to the deformation of thin objects such as paper and sheet metals around one axis by investigating the cross section perpendicular to the axis. Let us measure the deformation of two sheets of copy paper of $92(\mu m)$ thick shown in Figure 6-(a) and (b), respectively.

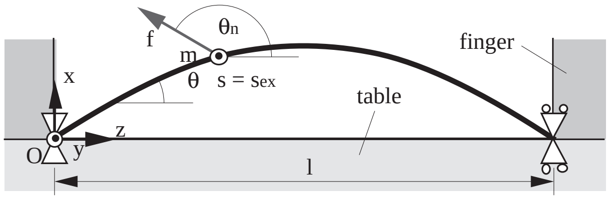
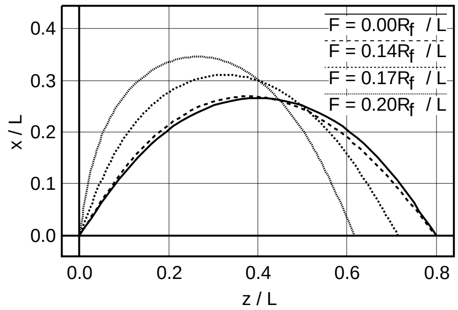
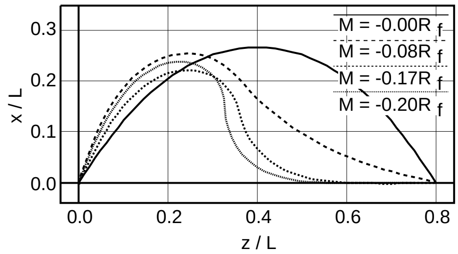


Figure 4: String reduced beforehand by fingers



(a) external force



(b) external moment

Figure 5: Deformation of string due to external force and moment

Figure 6-(a) shows a rectangle of $200(mm)$ long and $30(mm)$ wide. The bend rigidity R_f and the weight D per unit length of this paper are $10^4(gw \cdot mm^2)$ and $2 \times 10^{-3}(gw/mm)$, respectively. This paper is deformed so that the distance l be 180, 140, and $70(mm)$. In the computation, we assume that angles $\theta(0)$ and $\theta(L)$ are equal to zero. The difference between the computed values and experimental values along z-axis is 11(mm) at most. The ratio of the difference to the length of the paper is approximately 6%. The difference between the computed shapes and the measurement values results from the discrepancy between the given values and the actual values of angles $\theta(0)$ and $\theta(L)$. From the measurement values, we estimate that angles $\theta(0)$ and $\theta(L)$ are actually equal to $10°$ and $0°$, respectively. The computed values using the estimated angles are illustrated in Figure 7. The differ-

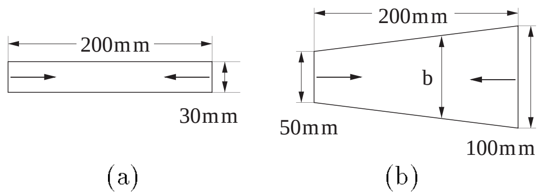


Figure 6: Experimental paper sheets

ence between the computed values and experimental values along z-axis is 2(mm) at most. Namely, the ratio of the difference to the paper length is reduced to 1%.

Figure 6-(b) shows a trapezoid of $200(mm)$ long with a left side $50(mm)$ long and a right side $100(mm)$ long. The bend rigidity R_f and the weight D of this paper can be given by $330b(gw \cdot mm^2)$ and $7b \times 10^{-5}(gw/mm)$, where b denotes the width of the paper. Note that the width b, which is given by $50 + s/4$, depends upon variable s. Thus, the bend rigidity and the weight vary according to variable s. The proposed method has a capability of computing the deformation in the case where the bend rigidity or the weight per unit length varies. Let us reduce this paper so that the distance l is equal to $160(mm)$. Without using estimated values of endpoints, the difference between the computed values and experimental values along z-axis is 8(mm) at most. The computed values using the estimated angles are illustrated in Figure 8. Note that the deformed shape of the object is unsymmetrical due to the ununiformity of the bend rigidity and the weight per unit length. This figure demonstrates the proposed method can compute unsymmetrical shape correctly. The difference between the computed values and experimental values along z-axis is 2(mm) at most. Namely, the ratio of the difference to the paper length is reduced to 1%.

6 Concluding Remarks

An analytical approach to the formulation of deformable strings in three-dimensional space has been developed based on the physical properties of the strings. First, we showed that the relationship between a natural shape of a string and its deformed shape should be represented in order to describe the string deformation. One generalized coordinate system was introduced so that the string deformation can be described appropriately. Secondly, internal energy and geometric constraints of a string were formulated using the introduced coordinates. It turned out that not only equational constraints resulting from predefined condition on the string motion but inequality constrains resulting from unidirectional nature of mechanical contacts are imposed on the string. Next, a procedure to compute the string deformation has been developed by applying nonlinear programming techniques. Some numerical examples and experimental results have demonstrated the effectiveness of the proposed approach.

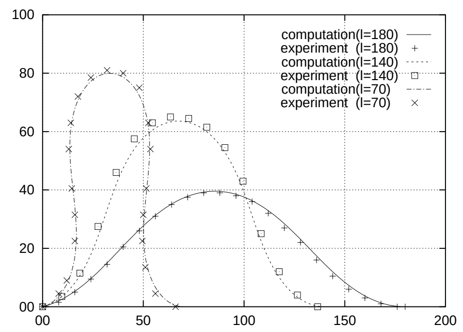


Figure 7: Computed deformed shape and measured deformed shape of rectangle paper

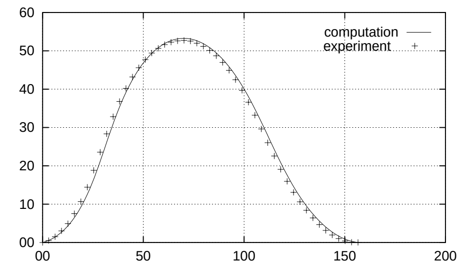


Figure 8: Computed deformed shape and measured deformed shape of trapezoid paper

References

[1] Taylor, P. M. et al., *Sensory Robotics for the Handling of Limp Materials*, Springer-Verlag, 1990

[2] Fung, Y. C., *Foundations of Solid Mechanics*, Prentice-Hall, 1965

[3] Rogers, D. F. and Adams, J. A., *Mathematical Elements for Computer Graphics*, McGraw-Hill, 1976

[4] Weil, J., *The Synthesis of Cloth Objects*, Computer Graphics, Vol 20, No.4, pp.49-54, 1986

[5] Terzopoulos, D. et al., *Elastically Deformable Models*, Computer Graphics, Vol 21, No.4, pp.205-214, 1987

[6] Mäntylä, M., *An Introduction to Solid Modeling*, Computer Science Press, 1988

[7] Crandall, S. H., Karnopp, D. C., Kurts, E. F., and Pridmore-Brown, D. C., *Dynamics of Mechanical and Electromechanical Systems*, McGraw-Hill, 1968

[8] Auslander, L., *Differential Geometry*, Harper International Edition, 1977

[9] Goldstein, H., *Classical Mechanics*, Addison-Wesley, 1980

[10] Elsgolc, L. E., *Calculus of Variations*, Pergamon Press, 1961

[11] Avriel, M., *Nonlinear Programming: Analysis and Methods*, Prentice-Hall, 1976