レオロジー変形の動的モデリング

Dynamic Modeling of Rheological Deformation

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We describe continua modeling of a rheologically deformable object. Rheological 2D deformation is formulated based on continua modeling. We show a simple simulation to demonstrate the modeling capability.

Key Words: rheology, modeling, deformation, continua, dynamic

1. Introduction

Most food and biological tissue show rheological nature in their deformation. Modeling and identification of these rheologically deformable objects are needed in virtual reality, especially, surgery simulation and mastication simulation. We have applied a particle-based approach to the modeling of rheological objects [(1)]. Unfortunately, physical meaning of model parameters is unclear in the particle-based approach, resulting the difficulty in identification of model parameters. Note that continua modeling stands on a clear foundation. In this paper, we apply the continua modeling to 2D rheological deformation to build a dynamic model of a rheological object.

2. Rheological objects

Objects deform in response to forces applied to the objects. Objects can be categorized into three groups with respect to their deformation. Assume that a natural shape of an object is as given in Figure 1-(a). On applying external forces, the object deforms as in Figure 1-(b). Let us release the applied force and examine the stable shape after the release. Deformation of *viscoelastic objects* is completely lost and their stable shape coincides with their natural shape, as illustrated in Figure 1-(c). Namely, viscoelastic objects have no *residual deformation*. Deformation of *plastic objects* completely remains and their stable shape coincides with their deformed shape under the applied forces, as shown in Figure 1-(d). Namely, plastic objects have no *bouncing deformation*. Objects with residual deformation and bouncing deformation are referred to as *rheological objects*. Deformation of rheological objects is partially lost after the applied forces are released, as illustrated in Figure 1-(e). Various objects including foods and tissues are categorized into rheological objects.

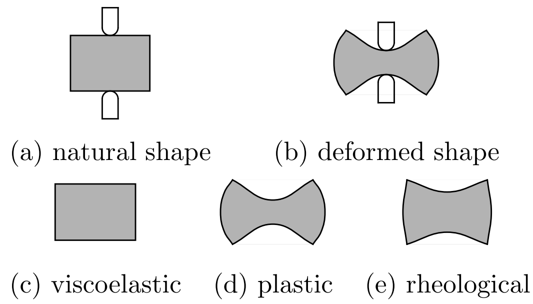


Fig.1 Viscoelastic object, plastic object, and rheological object

3. Dynamic modeling of 2D rheological object

Let $\boldsymbol{\sigma}$ be a pseudo stress vector and $\boldsymbol{\varepsilon}$ be a pseudo strain vector. Stress-strain relationship of 2D rheological deformation is formulated as follows:

$$\boldsymbol{\sigma}(t) = \int_0^t R(t-t')\,\dot{\boldsymbol{\varepsilon}}(t')\,\mathrm{d}t', \tag{1}$$

where 3×3 matrix $R(t-t')$ is referred to as a *relaxation matrix*, which determines the nature of a 2D rheological deformation. The relaxation matrix of 2D isotropic rheological deformation is formulated as

$$R(t-t') = r_\lambda(t-t')I_\lambda + r_\mu(t-t')I_\mu \tag{2}$$

where

$$\begin{aligned} r_\lambda(t-t') &= \lambda_{\rm ela}\exp\left\{-\frac{\lambda^{\rm ela}}{\lambda^{\rm vis}}(t-t')\right\}, \\ r_\mu(t-t') &= \mu_{\rm ela}\exp\left\{-\frac{\mu^{\rm ela}}{\mu^{\rm vis}}(t-t')\right\}. \end{aligned}$$

Elasticity of the object is specified by two elastic moduli $\lambda^{\rm ela}$ and $\mu^{\rm ela}$ while its viscosity is specified by two viscous moduli $\lambda^{\rm vis}$ and $\mu^{\rm vis}$. Matrices I_λ and I_μ are matrix representations of isotropic tensors, which are given as follows in 2D deformation:

$$I_\lambda = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_\mu = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The stress-strain relationship can be converted into a relationship between a set of forces applied to nodal points and a set of displacements of the points. Let $\boldsymbol{u}_N$ be a set of displacements of nodal points. Let J_λ and J_μ are connection matrices, which can be geometrically determined by object coordinate components of nodal points. Replacing I_λ by J_λ, I_μ by J_μ, and $\boldsymbol{\varepsilon}$ by $\boldsymbol{u}_N$ in the stress-strain relationship (1) of a rheological object yields a set of rheological forces applied to nodal points as follows:

$$\text{rheological force} = J_\lambda \boldsymbol{w}_\lambda + J_\mu \boldsymbol{w}_\mu \tag{3}$$

where

$$\boldsymbol{w}_\lambda = \int_0^t \lambda_{\rm ela}\exp\left\{-\frac{\lambda^{\rm ela}}{\lambda^{\rm vis}}(t-t')\right\}\dot{\boldsymbol{u}}_N(t')\,\mathrm{d}t',$$

$$\boldsymbol{w}_\mu \quad = \quad \int_0^t \mu_{\rm ela} \exp\left\{ -\frac{\mu^{\rm ela}}{\mu^{\rm vis}}(t-t') \right\} \, \dot{\boldsymbol{u}}_N(t') \, {\rm d}t'.$$

Let M be an inertia matrix and $\boldsymbol{f}$ be a set of external forces applied to nodal points. Let us describe a set of geometric constraints imposed on the nodal points by $A^T \boldsymbol{u}_N = \boldsymbol{b}$. The number of columns of matrix A is equal to the number of geometric constraints. Let $\boldsymbol{\lambda}$ be a set of constraint forces corresponding to the geometric constraints. A set of dynamic equations of nodal points is then given by

$$-(J_\lambda \boldsymbol{w}_\lambda + J_\mu \boldsymbol{w}_\mu) + \boldsymbol{f} + A\boldsymbol{\lambda} - M\ddot{\boldsymbol{u}}_N = \mathbf{o}.$$

Applying the constraint stabilization method [2] to the constraints specified by angular velocity ω, system dynamic equations are described as follows:

$$\begin{aligned}
\dot{\boldsymbol{u}}_N &= \boldsymbol{v}_N, \\
M\dot{\boldsymbol{v}}_N - A\boldsymbol{\lambda} &= -J_\lambda \boldsymbol{w}_\lambda - J_\mu \boldsymbol{w}_\mu + \boldsymbol{f}, \\
-A^T \dot{\boldsymbol{v}}_N &= A^T(2\omega \boldsymbol{v}_N + \omega^2 \boldsymbol{u}_N), \\
\dot{\boldsymbol{w}}_\lambda &= -\frac{\lambda^{\rm ela}}{\lambda^{\rm vis}} \boldsymbol{w}_\lambda + \lambda^{\rm ela} \boldsymbol{v}_N, \\
\dot{\boldsymbol{w}}_\mu &= -\frac{\mu^{\rm ela}}{\mu^{\rm vis}} \boldsymbol{w}_\mu + \mu^{\rm ela} \boldsymbol{v}_N.
\end{aligned}$$

Consequently,

$$\begin{bmatrix} I & & & & \\ & M & -A & & \\ & -A^T & & & \\ & & & I & \\ & & & & I \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{u}}_N \\ \dot{\boldsymbol{v}}_N \\ \boldsymbol{\lambda} \\ \dot{\boldsymbol{w}}_\lambda \\ \dot{\boldsymbol{w}}_\mu \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_N \\ -J_\lambda \boldsymbol{w}_\lambda - J_\mu \boldsymbol{w}_\mu + \boldsymbol{f} \\ A^T(2\omega \boldsymbol{v}_N + \omega^2 \boldsymbol{u}_N) \\ -\frac{\lambda^{\rm ela}}{\lambda^{\rm vis}} \boldsymbol{w}_\lambda + \lambda^{\rm ela} \boldsymbol{v}_N \\ -\frac{\mu^{\rm ela}}{\mu^{\rm vis}} \boldsymbol{w}_\mu + \mu^{\rm ela} \boldsymbol{v}_N \end{bmatrix}. \quad (4)$$

Note that the above linear equation is solvable since the matrix is regular, implying that we can sketch $\boldsymbol{u}_N$, $\boldsymbol{v}_N$, $\boldsymbol{w}_\lambda$, and $\boldsymbol{w}_\mu$ using numerical solver such as the Euler method or the Runge-Kutta method.

4. Simulation

Let us apply the dynamic model of 2D rheological deformation to a 2D beam illustrated in Figure 2. The beam involves 10 nodal points and 8 triangles. Edge P_0P_5 is affixed on a wall. Uniform pressure $\boldsymbol{P} = [P_x, P_y]^T$ is applied over an edge P_4P_9. Values of elastic moduli are $\lambda_{\rm ela} = 7.0$, $\mu_{\rm ela} = 5.0$, values of viscous moduli are $\lambda_{\rm vis} = 4.0$, $\mu_{\rm vis} = 2.0$, and area density is given by $\rho = 0.2$. Pressure $\boldsymbol{P} = [10, 0]^T$ is applied during the first 1 second. After 1 second, no pressure is applied on the right edge.

Figure 3 shows a successive shape of the deforming object. As shown in the figure, the beam extends during the first 1 second and shrinks after the applied pressure is released. This implies that the simulation describes the rheological deformation of the beam. Deformation along the vertical axis is caused by non-uniform arrangement of triangles. Residual forces $\boldsymbol{w}_\lambda$ and $\boldsymbol{w}_\mu$ converge to zero as plotted in Figure 4.

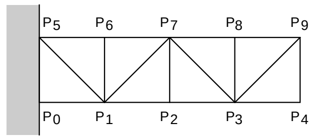
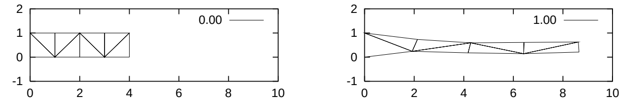
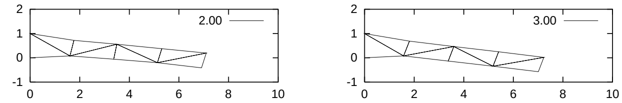


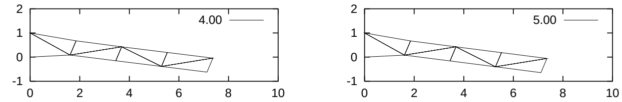
Fig.2 Two-dimensional rheological beam



(a) 0[s] (b) 1[s]

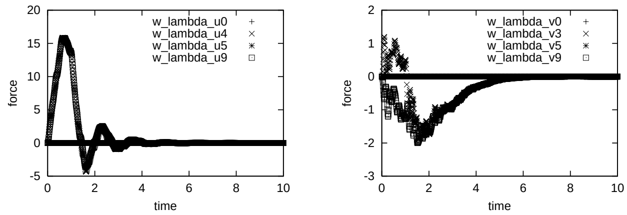


(c) 2[s] (d) 3[s]

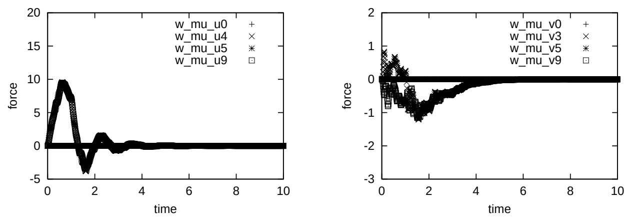


(e) 4[s] (f) 5[s]

Fig.3 Simulation of 2D rheological deformation



(a) $\boldsymbol{w}_\lambda$



(b) $\boldsymbol{w}_\mu$

Fig.4 Residual forces

5. Conclusion

We have applied the continua modeling to rheological deformation and have built a dynamic model of a 2D rheological object. Experimental evaluation will be studied soon.

References

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