Modeling of Linear and Belt Object Deformation Based on Differential Geometry

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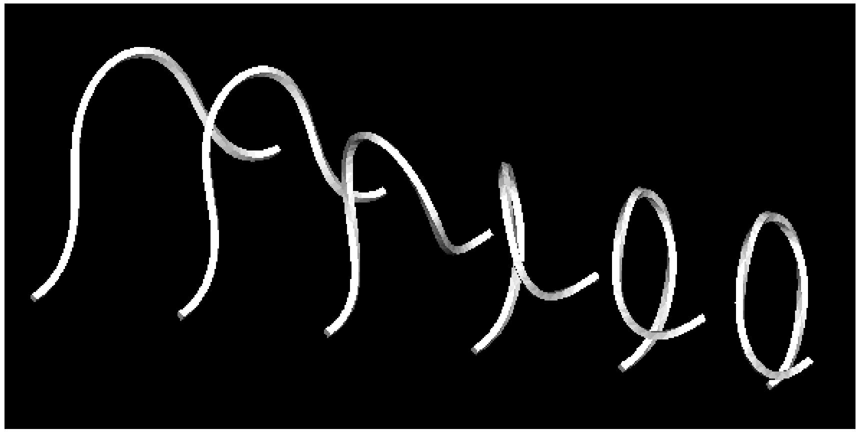
Agenda

- Introduction
- Modeling of Linear Object Deformation
- Application to Linear Object Structure
- Modeling of Belt Object Deformation
- Conclusions



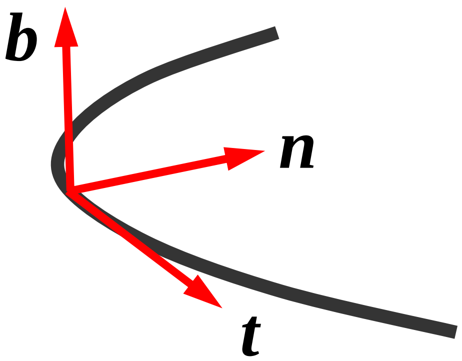
Manipulation of Flexible Linear/Belt Objects

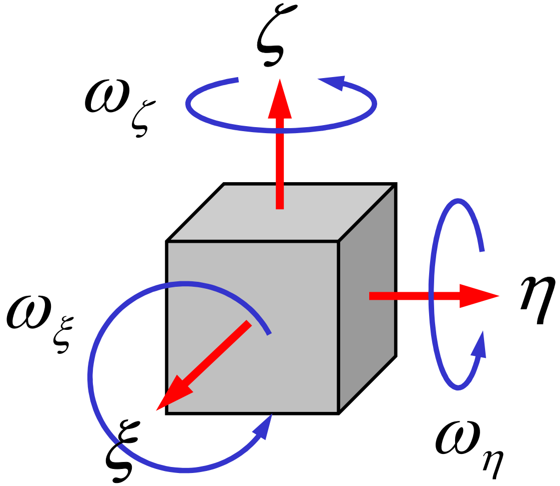
Wire harness

Flexible printed circuit board

A modeling of linear/belt object deformation is required for planning of manipulative operations and their execution by a mechanical system.

Frenet-Serret Formulas





$$\begin{bmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}$$

Frenet-Serret formulas

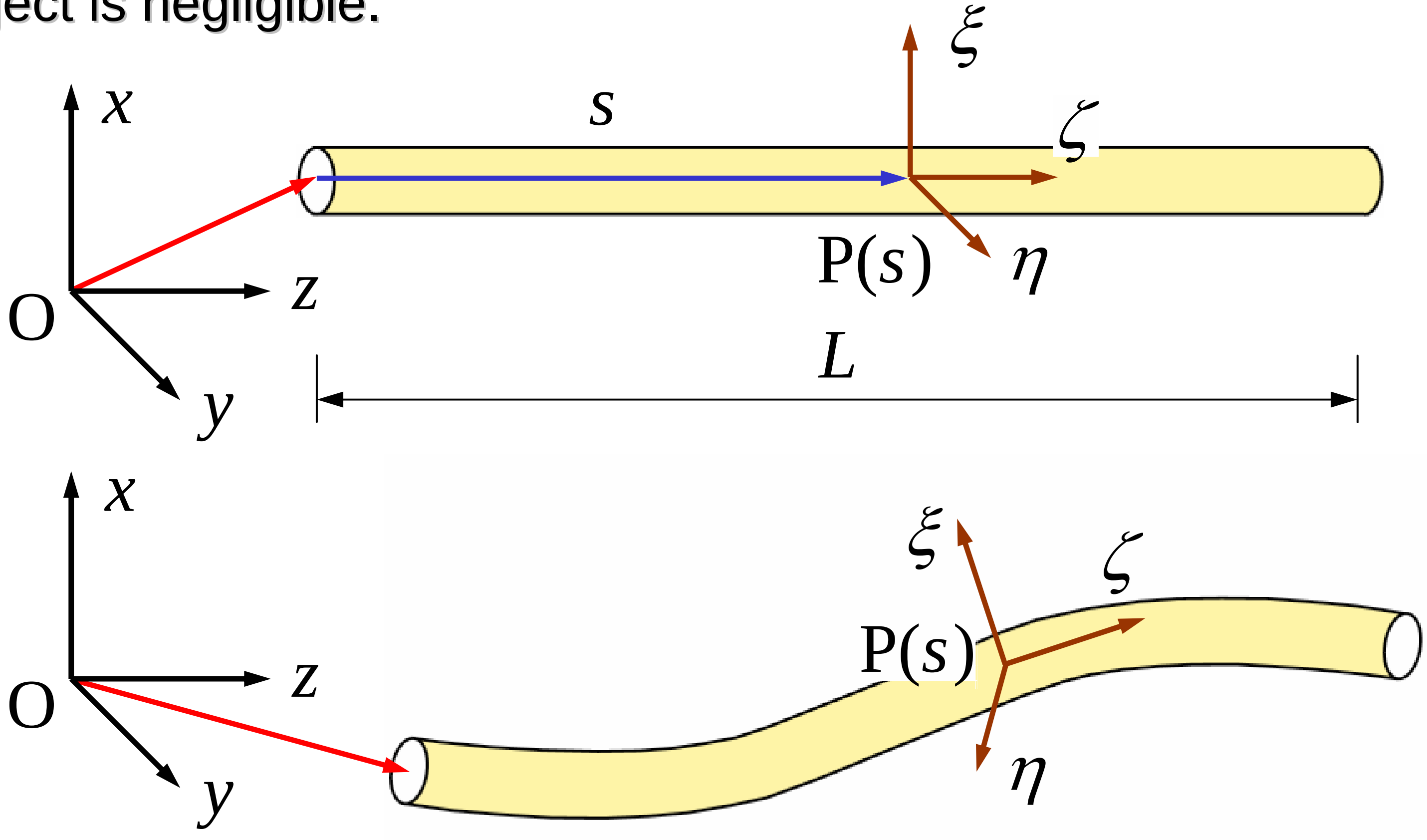
$$\begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = \begin{bmatrix} 0 & \omega_\zeta & -\omega_\eta \\ -\omega_\zeta & 0 & \omega_\xi \\ \omega_\eta & -\omega_\xi & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$$

Angular verocities of rigid body

Modeling of Linear Object Deformation

Assumption :

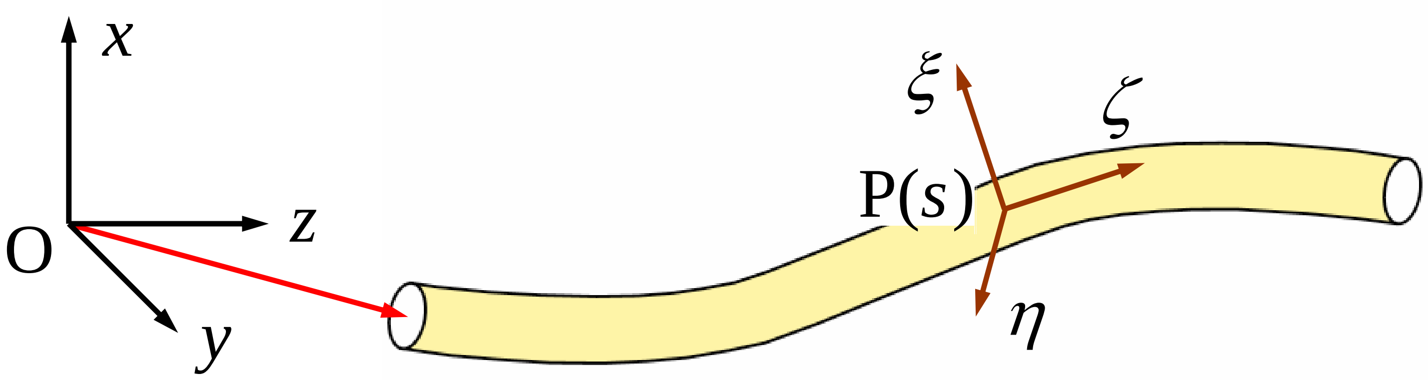
Deformation in any direction perpendicular to the central axis of a linear object is negligible.



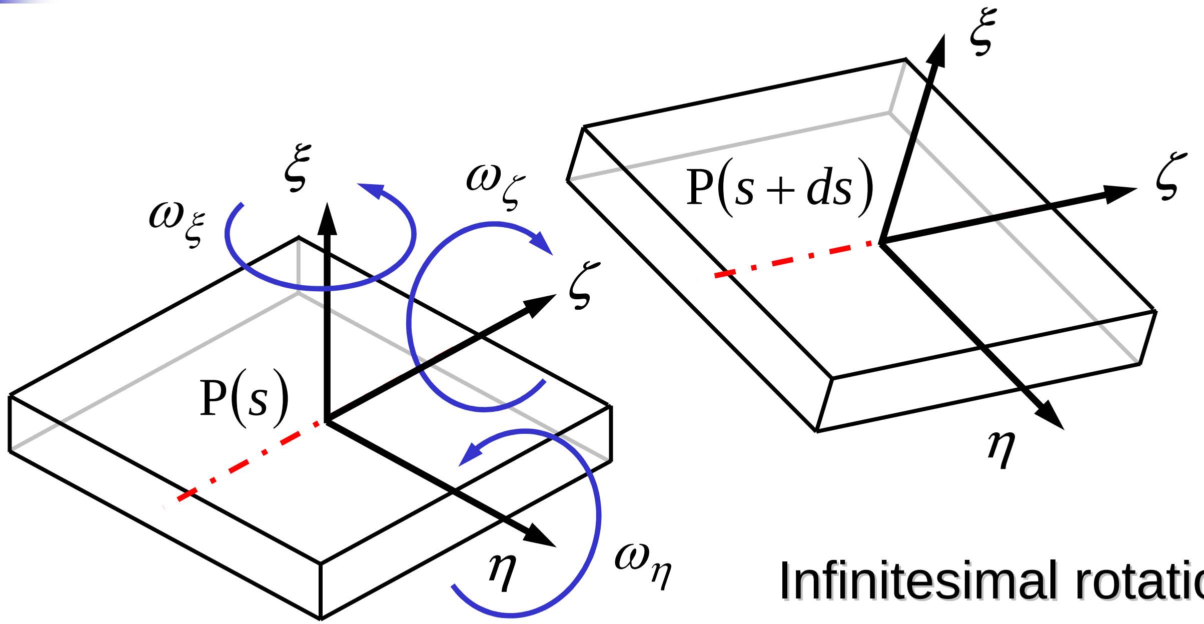
Rotation Matrix

Rotation matrix :

$$A=\begin{bmatrix} \cos\theta\cos\phi\cos\psi-\sin\phi\sin\psi & -\cos\theta\cos\phi\sin\psi-\sin\phi\cos\psi & \sin\theta\cos\phi \\ \cos\theta\sin\phi\cos\psi+\cos\phi\sin\psi & -\cos\theta\sin\phi\sin\psi+\cos\phi\cos\psi & \sin\theta\sin\phi \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$



Rotation of Object Coordinate System



Infinitesimal rotational angles :

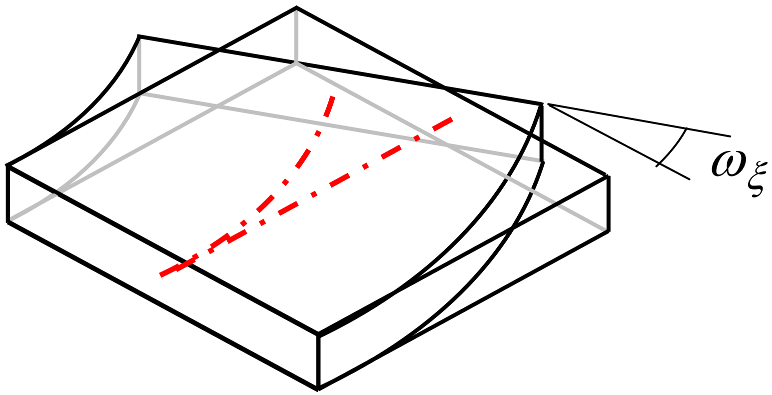
$$\frac{d}{ds}\begin{bmatrix}\xi(s)\\ \eta(s)\\ \zeta(s)\end{bmatrix}=\begin{bmatrix}0 & \omega_\zeta & -\omega_\eta\\ -\omega_\zeta & 0 & \omega_\xi\\ \omega_\eta & -\omega_\xi & 0\end{bmatrix}\begin{bmatrix}\xi(s)\\ \eta(s)\\ \zeta(s)\end{bmatrix}$$

$$\omega_\xi = \frac{d\theta}{ds}\sin\psi - \frac{d\phi}{ds}\sin\theta\cos\psi$$

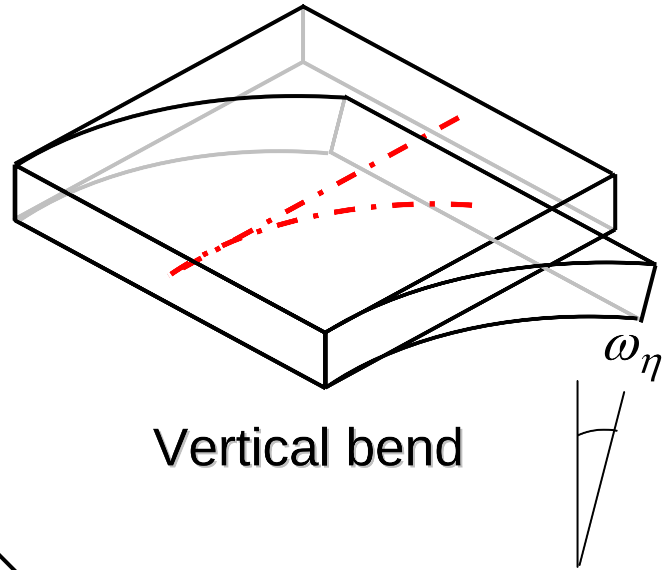
$$\omega_\eta = \frac{d\theta}{ds}\cos\psi + \frac{d\phi}{ds}\sin\theta\sin\psi$$

$$\omega_\zeta = \frac{d\phi}{ds}\cos\theta + \frac{d\psi}{ds}$$

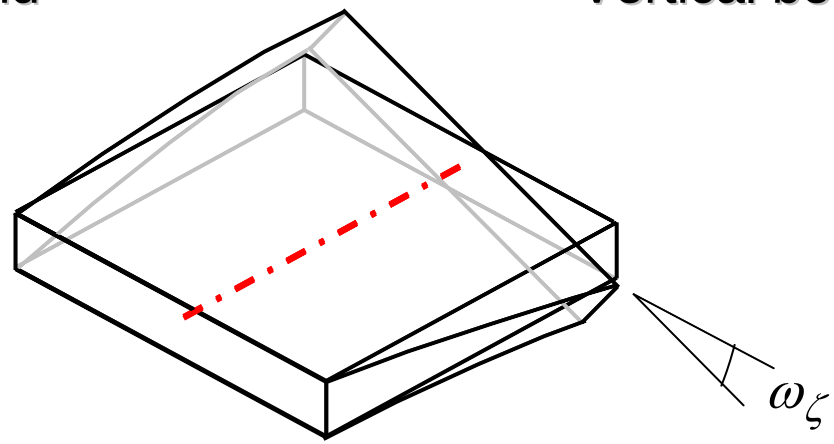
Infinitesimal Rotational Angles



Horizontal bend

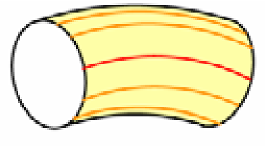


Vertical bend



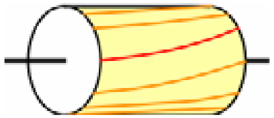
Twist

Curvature, Torsional Ratio, and Normal Strain



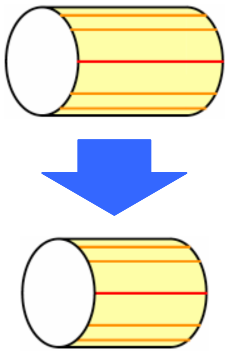
Curvature :

$$\kappa^2 = \omega_\xi{}^2 + \omega_\eta{}^2 = \left(\frac{\mathrm{d}\theta}{\mathrm{ds}}\right)^2 + \left(\frac{\mathrm{d}\phi}{\mathrm{ds}}\right)^2 \sin^2\theta$$



Torsional ratio :

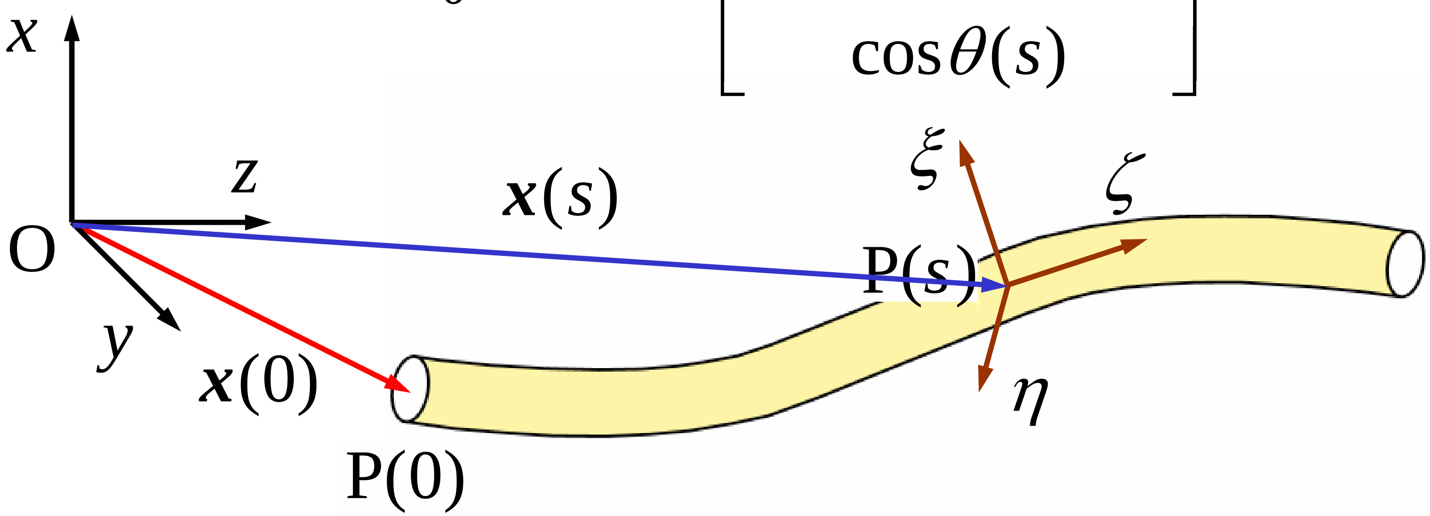
$$\omega^2 = \omega_\zeta{}^2 = \left(\frac{d\psi}{ds} + \frac{d\phi}{ds}\cos\theta\right)^2$$



Normal strain : $\varepsilon(s)$

Spatial Coordinates

$$\boldsymbol{x}(s) = \boldsymbol{x}(0) + \int_0^s \{1-\varepsilon(s)\} \begin{bmatrix} \sin\theta(s)\cos\phi(s) \\ \sin\theta(s)\sin\phi(s) \\ \cos\theta(s) \end{bmatrix} ds$$



The geometrical shape of a deformed linear object can be represented by four functions : $\phi(s), \theta(s), \psi(s), \varepsilon(s)$

Potential Energy

Variational principle in statics :

The potential energy of a linear object attains its minimum value in its stable deformed state under the imposed constraints.

Potential energy : $U = U_{\mathrm{flex}} + U_{\mathrm{tor}} + U_{\mathrm{ext}} + U_{\mathrm{grav}}$

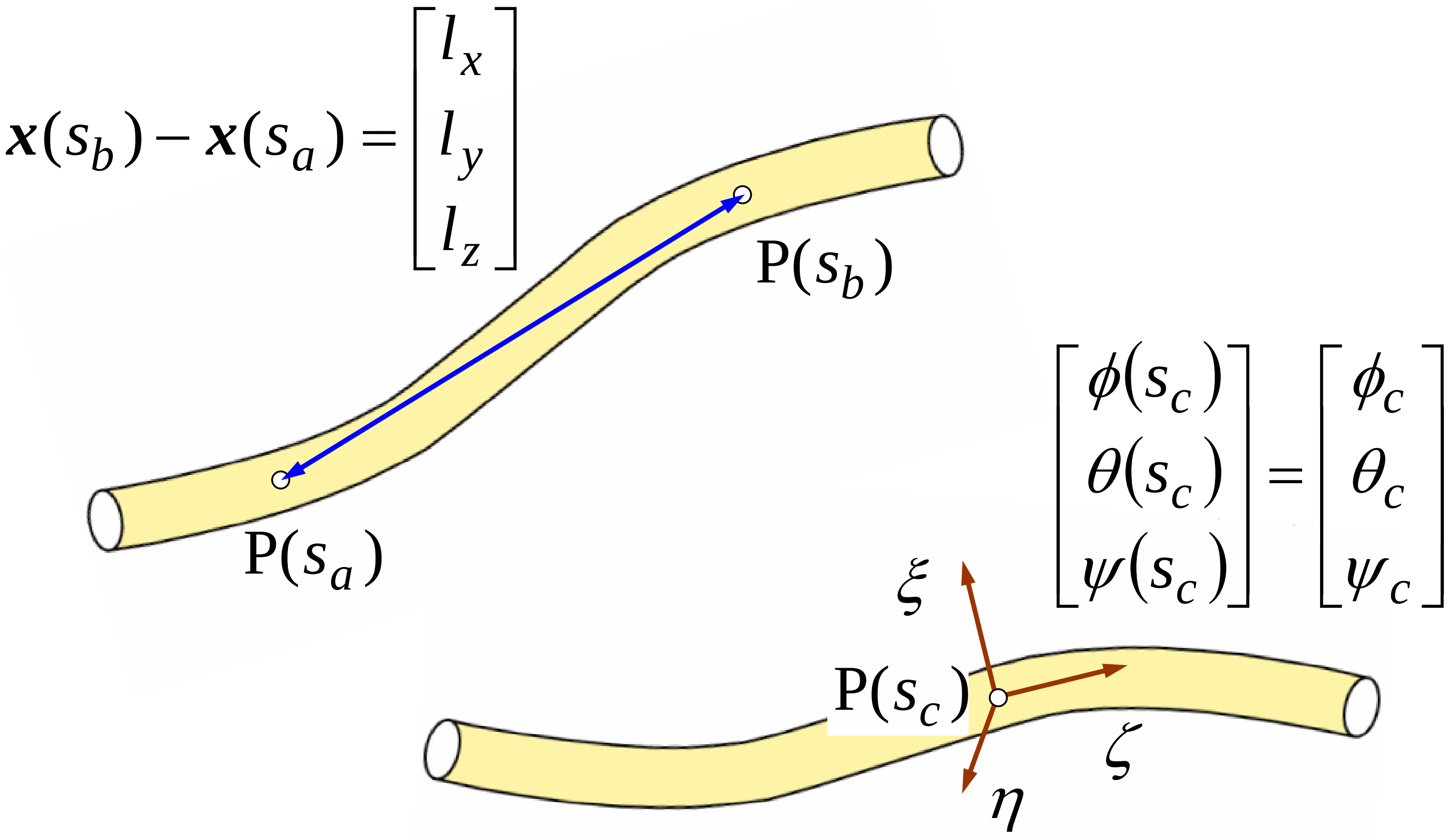
Flexural energy : $U_{\mathrm{flex}} = \int_0^L \frac{1}{2} R_f \kappa^2 ds$ R_f : Flexural rigidity

Torsional energy : $U_{\mathrm{tor}} = \int_0^L \frac{1}{2} R_t \omega^2 ds$ R_t : Torsional rigidity

Extensional energy : $U_{\mathrm{ext}} = \int_0^L \frac{1}{2} R_e \varepsilon^2 ds$ R_e : Extentional rigidity

Gravitational energy : $U_{\mathrm{grav}} = \int_0^L Dgxds$ D : Linear density

Positional/Orientational Constraints

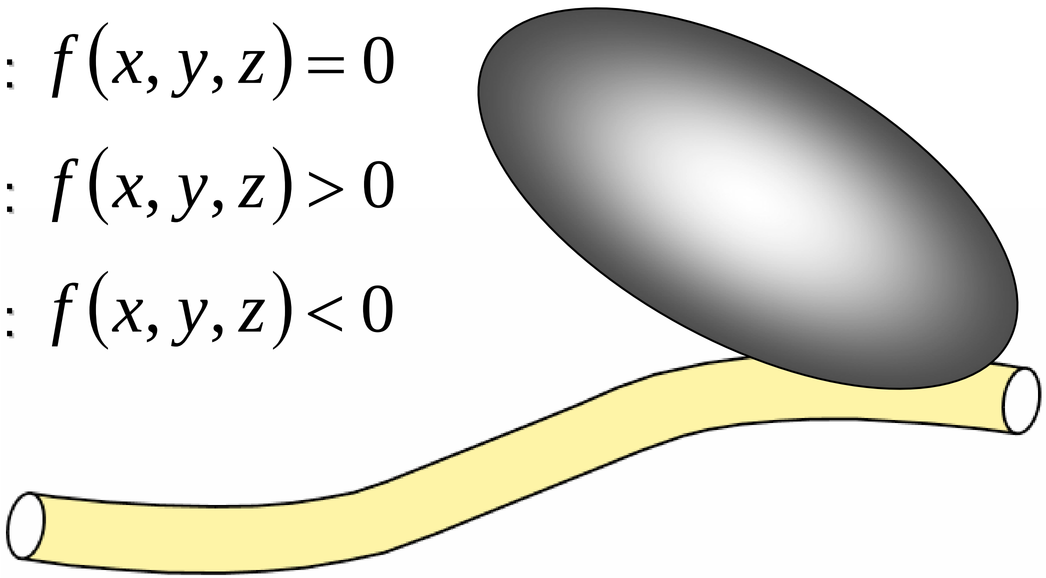


Consideration of Contact with Obstacles

Surface : $f(x, y, z) = 0$

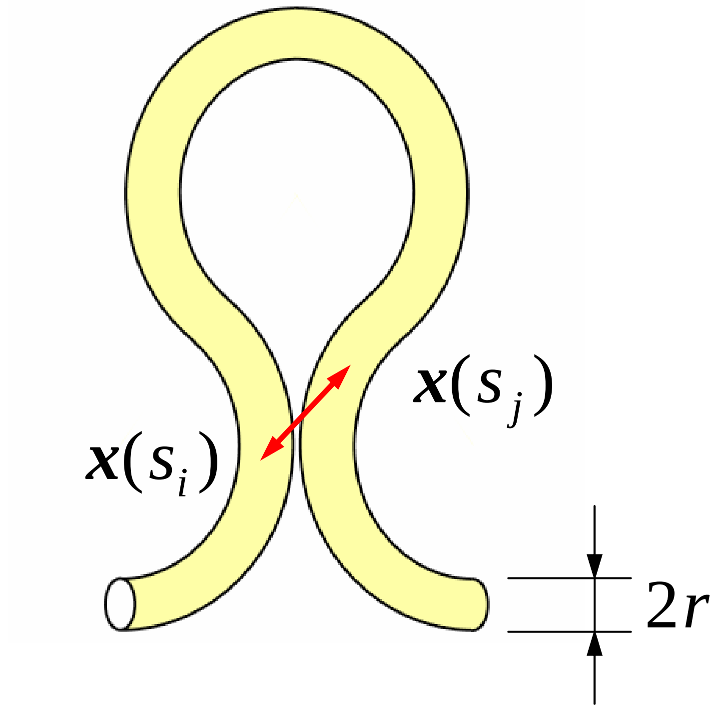
Inside : $f(x, y, z) > 0$

Outside : $f(x, y, z) < 0$



$$f(x(s), y(s), z(s)) \leq 0, \quad \forall s \in [0, L]$$

Consideration of Self-interaction



$$\left|\boldsymbol{x}(s_i) - \boldsymbol{x}(s_j)\right| \geq 2r, \quad \forall s_i, s_j \in [0, L], \text{ s.t. } \left|s_i - s_j\right| \geq 2r$$

The geometrical constraints imposed on a linear object are given by not only equational conditions but also inequality conditions.

Minimization Problem

$$\phi(s)=\sum_{i=1}^{n}a_i^{\phi}e_i(s),\ \ \theta(s)=\sum_{i=1}^{n}a_i^{\theta}e_i(s),$$

$$\psi(s)=\sum_{i=1}^{n}a_i^{\psi}e_i(s),\ \ \varepsilon(s)=\sum_{i=1}^{n}a_i^{\varepsilon}e_i(s)$$

$$\phi(s)=\boldsymbol{a}^{\phi}\cdot\boldsymbol{e}(s),\ \ \theta(s)=\boldsymbol{a}^{\theta}\cdot\boldsymbol{e}(s),\ \ \psi(s)=\boldsymbol{a}^{\psi}\cdot\boldsymbol{e}(s),\ \ \varepsilon(s)=\boldsymbol{a}^{\varepsilon}\cdot\boldsymbol{e}(s)$$

$$\boldsymbol{a}=\lfloor\boldsymbol{a}^{\phi}\quad\boldsymbol{a}^{\theta}\quad\boldsymbol{a}^{\psi}\quad\boldsymbol{a}^{\varepsilon}\rfloor$$

Minimize potential energy $U(\boldsymbol{a})$

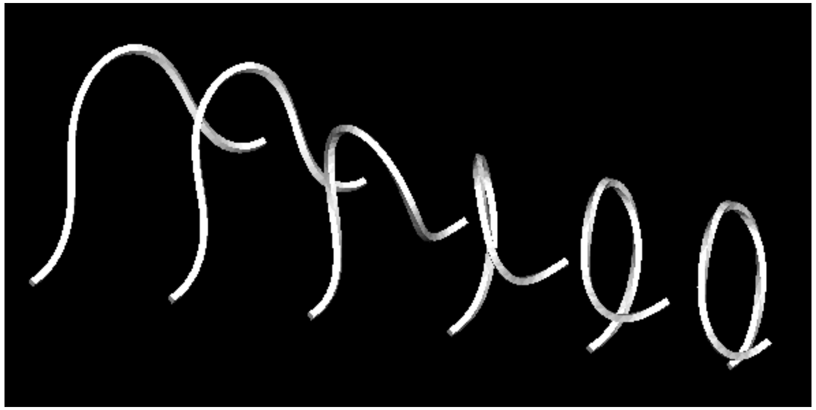
Subject to $f_j(\boldsymbol{a})=0 \quad (j=1,\cdots,J)$

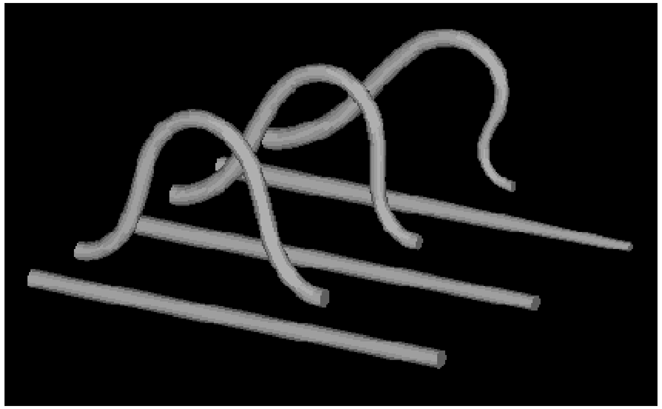
Positional/orientational constraints

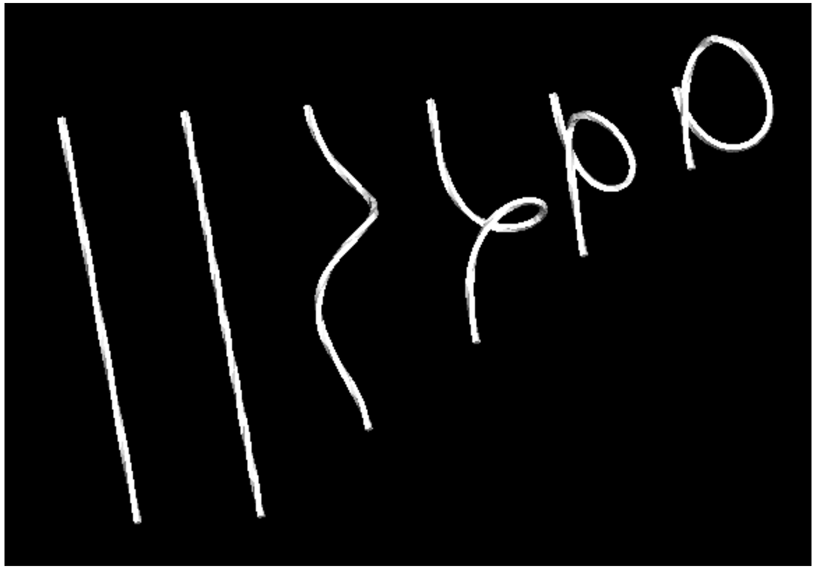
$g_k(\boldsymbol{a})\le 0 \quad (k=1,\cdots,K)$

Avoidance of (self-)interference

Computational Results







Basis functions:

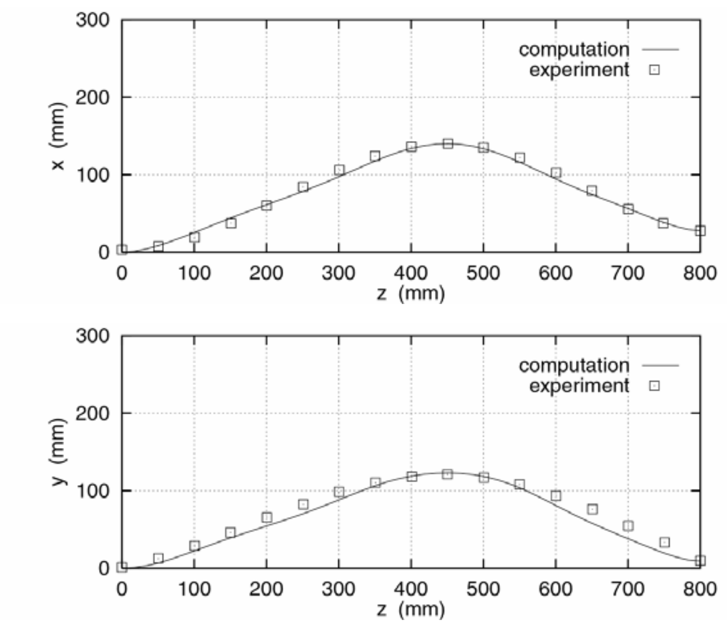
$$e_1 = 1,\ e_2 = s,$$

$$e_{2i+1} = \sin\frac{\pi i s}{L},$$

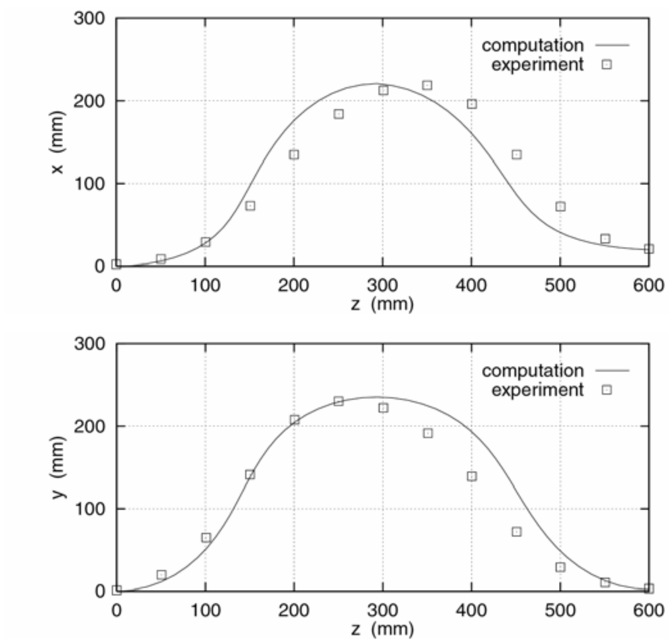
$$e_{2i+2} = \cos\frac{\pi i s}{L} \quad (i = 1, 2, 3, 4)$$

Experimental Verification (1)

Length	8.7×10^{2} [mm]
Flexural rigidity	6.6×10^{-4} [Nm^2]
Torsional rigidity	2.3×10^{-4} [Nm^2]
Weight per unit length	1.0×10^{-2} [N/m]

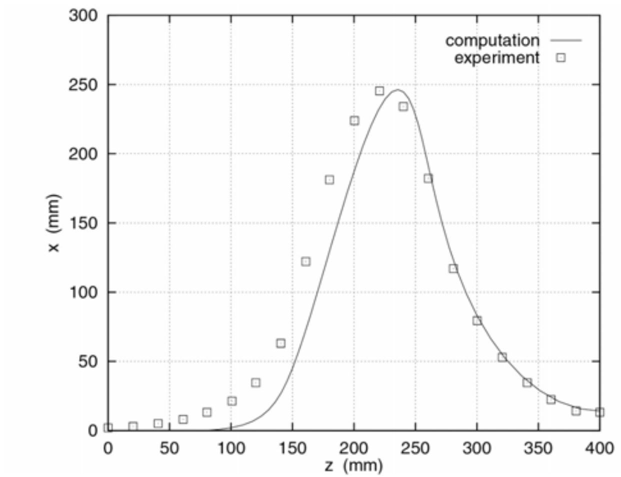


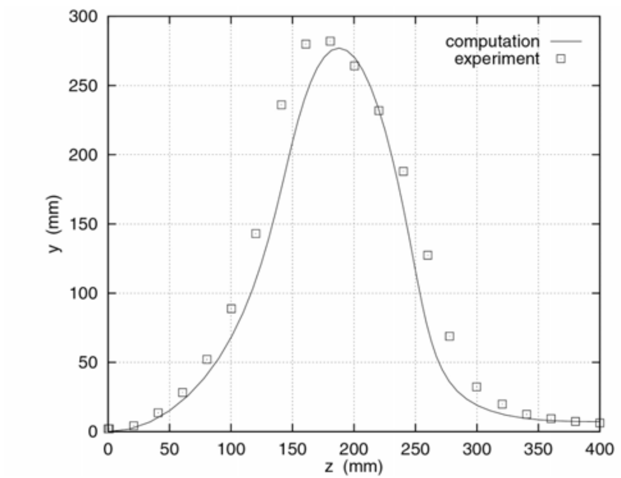
$l_z = 800\,[mm]$



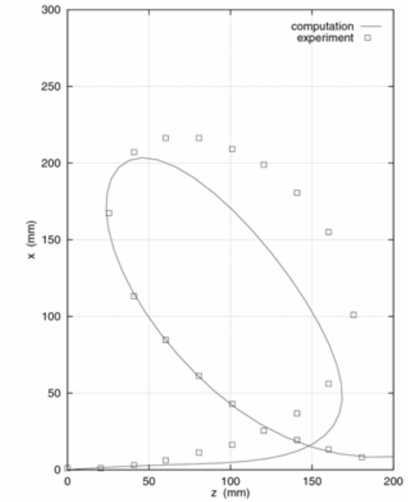
$l_z = 600\,[mm]$

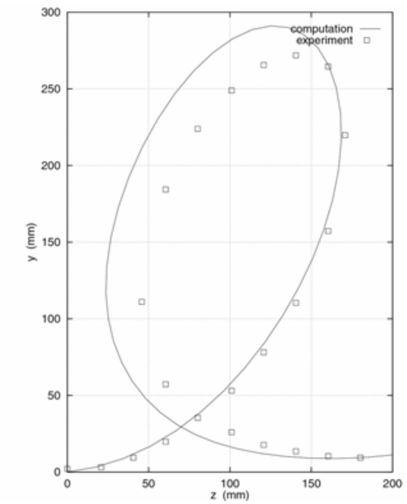
Experimental Verification (2)





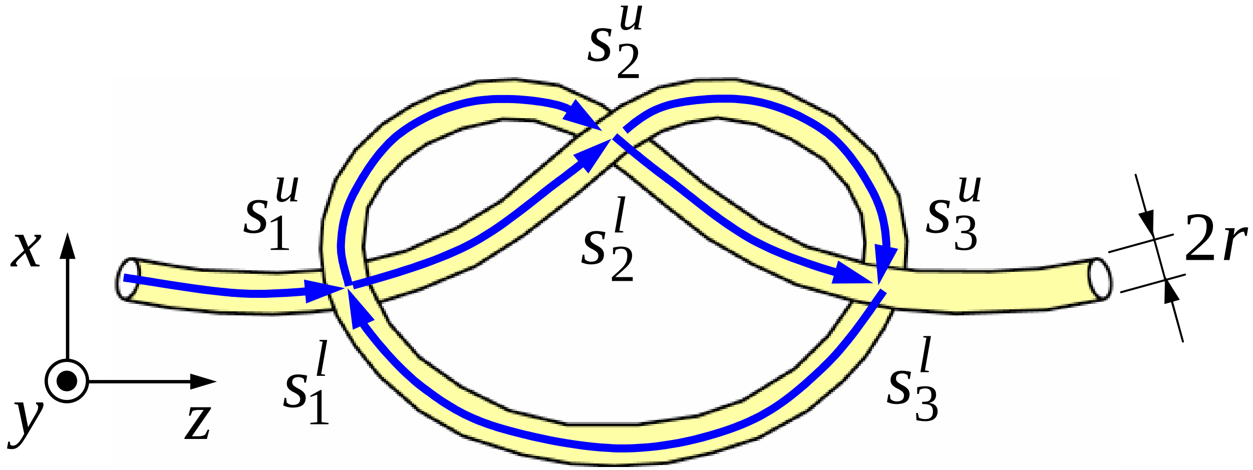
$l_z = 400\,[mm]$





$l_z = 200\,[mm]$

Knotted Shape of Linear Object



$$z(s_1^u) - z(s_1^l) = 0,\ z(s_2^u) - z(s_2^l) = 0,\ z(s_3^u) - z(s_3^l) = 0,$$

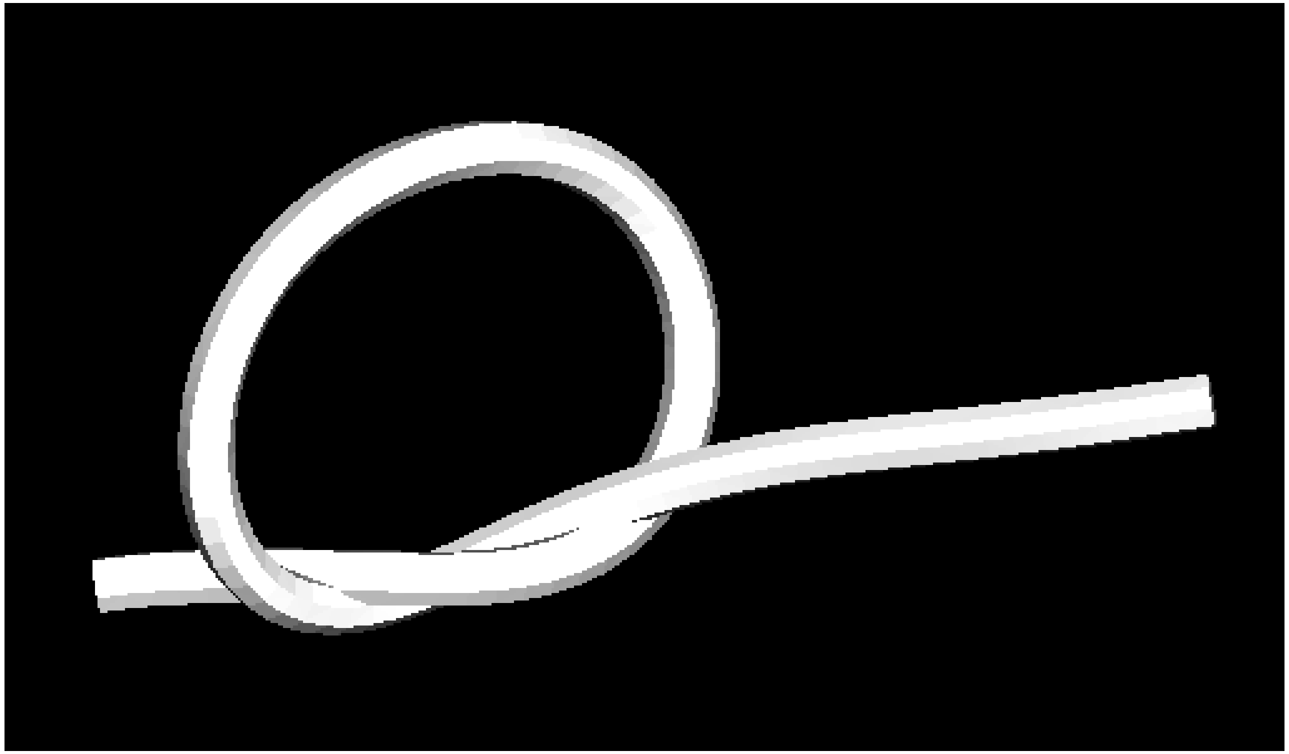
$$x(s_1^u) - x(s_1^l) = 0,\ x(s_2^u) - x(s_2^l) = 0,\ x(s_3^u) - x(s_3^l) = 0,$$

$$y(s_1^u) - y(s_1^l) = 2r,\ y(s_2^u) - y(s_2^l) = 2r,\ y(s_3^u) - y(s_3^l) = 2r,$$

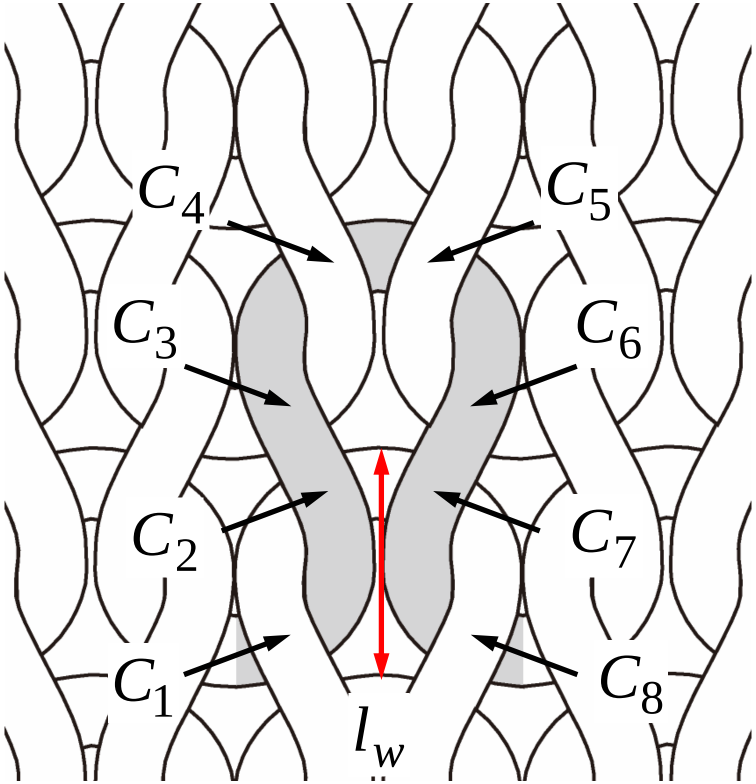
$$0 \le \underline{s_1^l < s_2^u < s_3^l < s_1^u < s_2^l < s_3^u} \le L$$

$$\boldsymbol{a} = \begin{bmatrix} \boldsymbol{a}^\phi & \boldsymbol{a}^\theta & \boldsymbol{a}^\psi & s_1^u & s_1^l & s_2^u & s_2^l & s_3^u & s_3^l \end{bmatrix}$$

Computational Result of Overhand Knot



Knitted Shape of Linear Objects

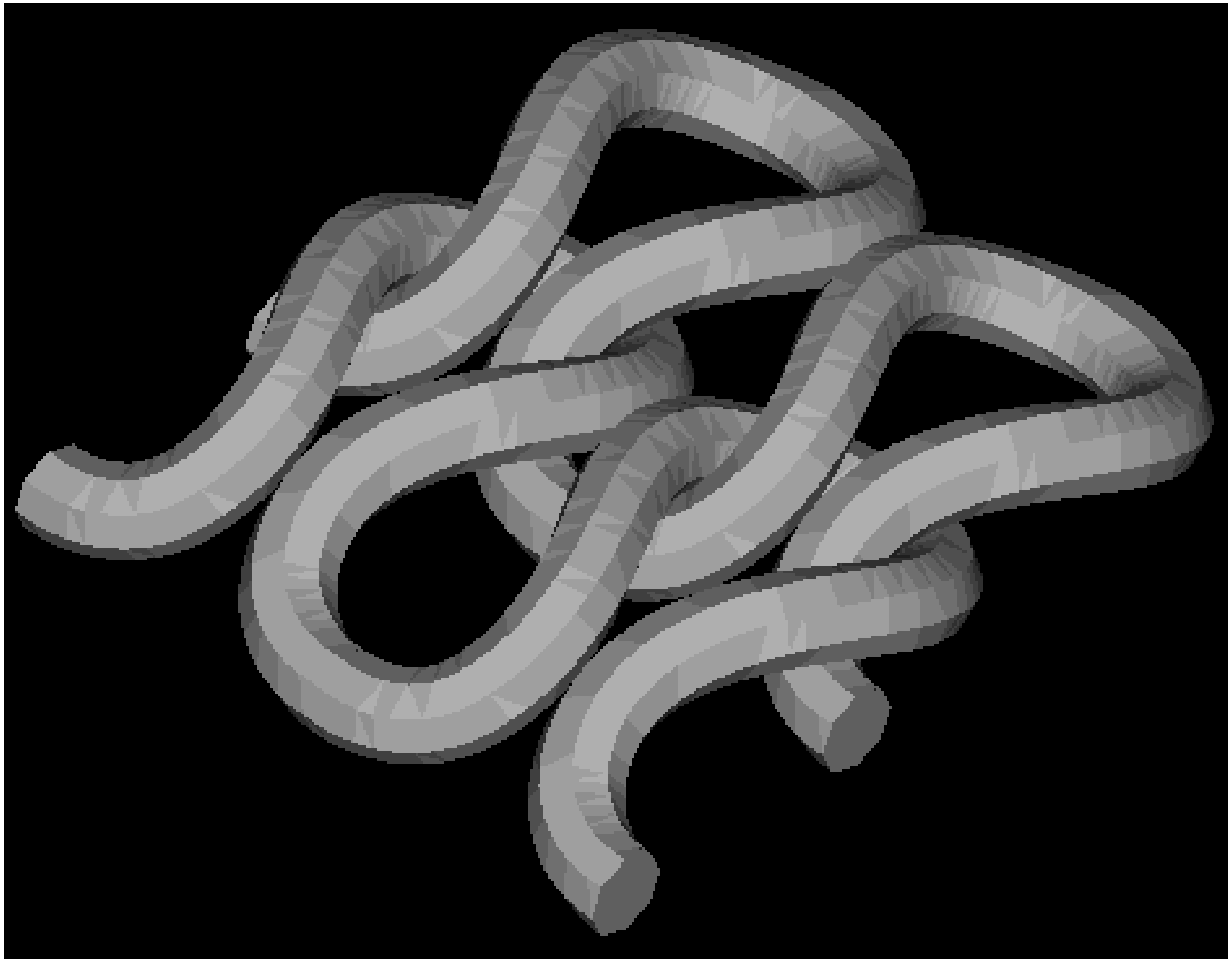


$$z(s_i) - z(s_{i+2}) = 0, \quad (i = 1, 2, 5, 6)$$
$$x(s_i) - x(s_{i-2}) = l_w, \quad (i = 3, 4)$$
$$x(s_i) - x(s_{i+2}) = l_w, \quad (i = 5, 6)$$
$$y(s_i) - y(s_{i-2}) = 2r, \quad (i = 3, 7)$$
$$y(s_i) - y(s_{i+2}) = 2r, \quad (i = 2, 6)$$
$$0 \le s_i < s_{i+1} \le L, \quad (i = 1, \cdots, 7)$$

Assumption :

The shape of the fabric can be represented by repetitions of the same shape of one loop.

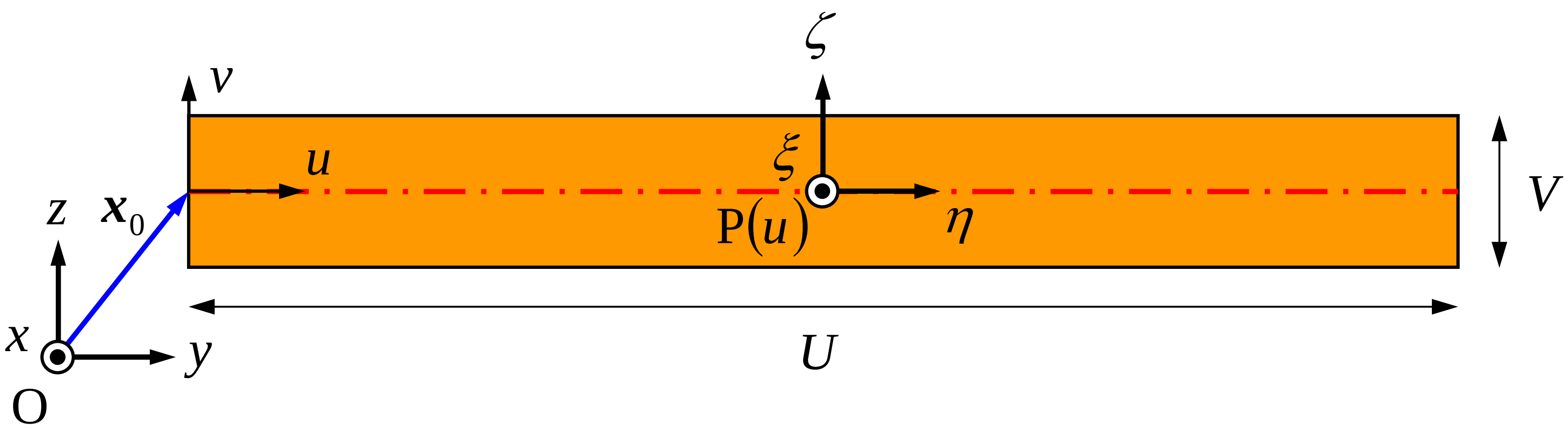
Computational Result of Plain Knitted Fabric



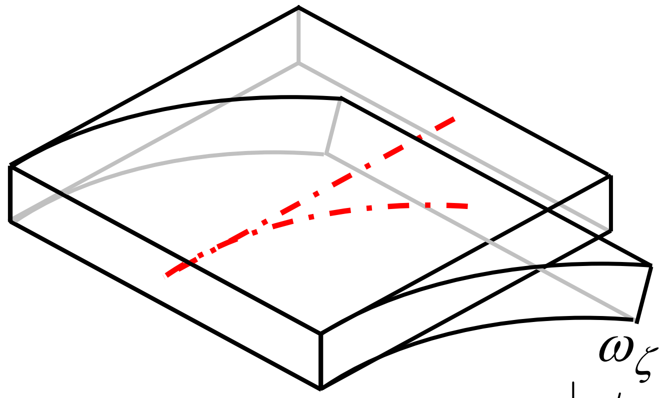
Modeling of Belt Object Deformation

Assumptions:

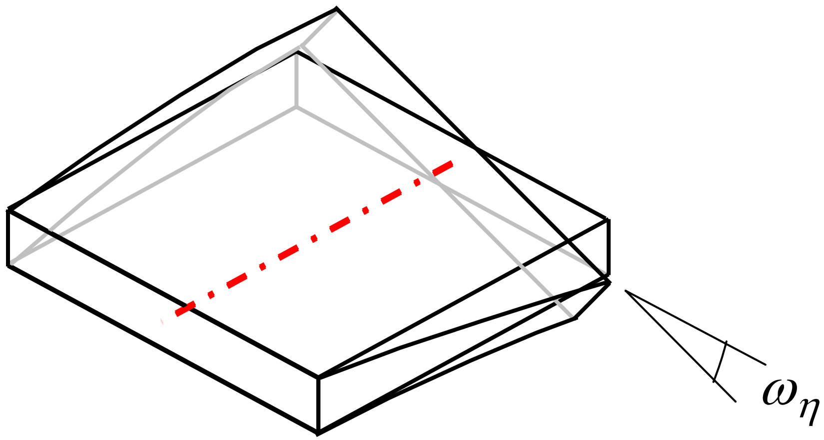
- A belt object is rectangular.
- The width of the object is sufficiently small compared to its length.
- The object is inextensible. Namely, it can be bent and twisted but cannot be expanded or contracted.
- Its both ends cannot be deformed because connectors are attached to the ends.



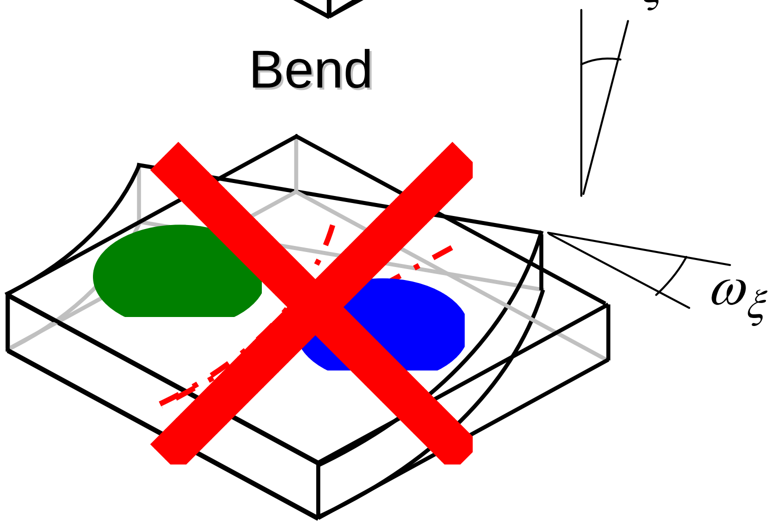
Infinitesimal Rotational Angles



Bend



Twist



Shape in *uv*-space

Assumption:

A belt object is inextensible.

In case of rectangular object:

$$\omega_\xi \equiv 0$$

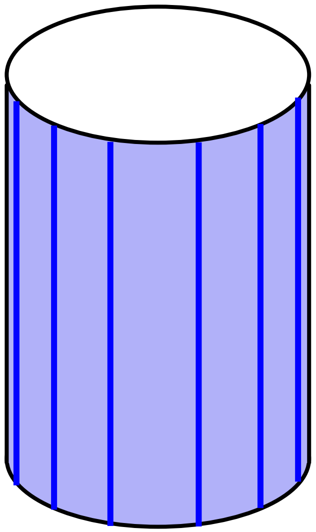
Developable Surfaces

Assumption:

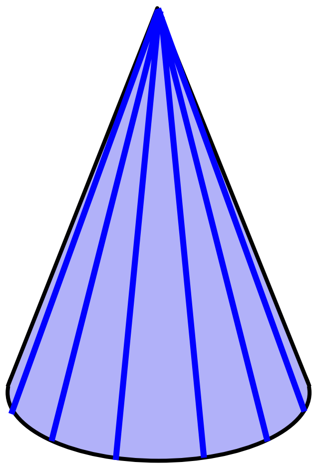
A belt object is inextensible. → Its surface is developable.

Developable surface :

- It can be generated by sweeping a straight line in 3D space.
- It includes straight lines.

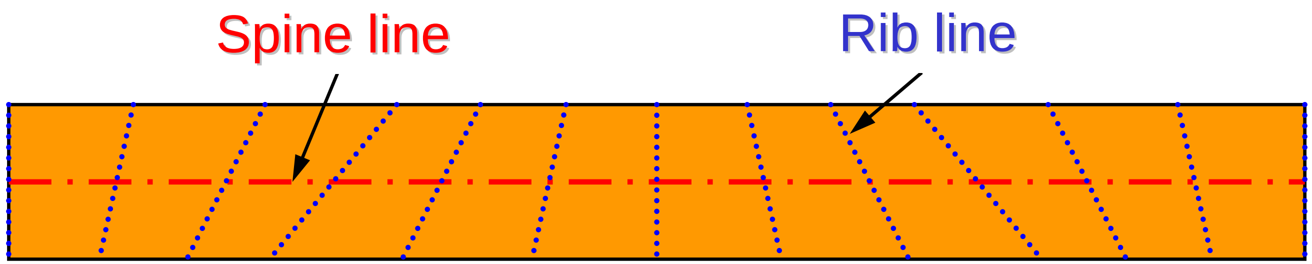


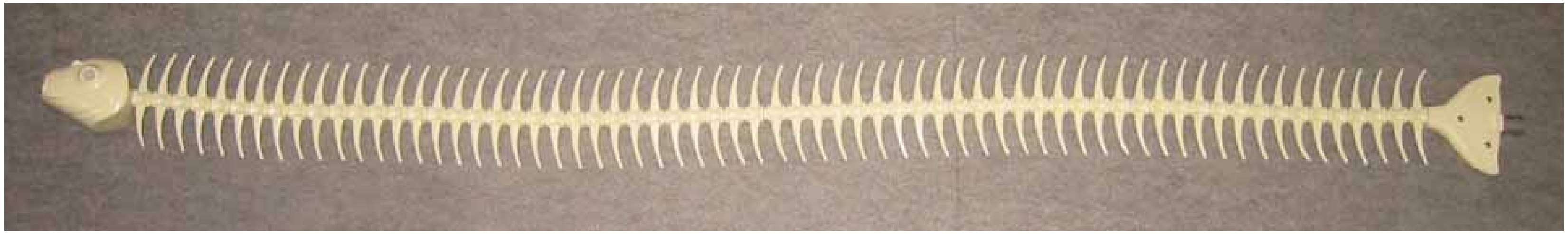
Cylindrical surface



Conic surface

Fishbone Model





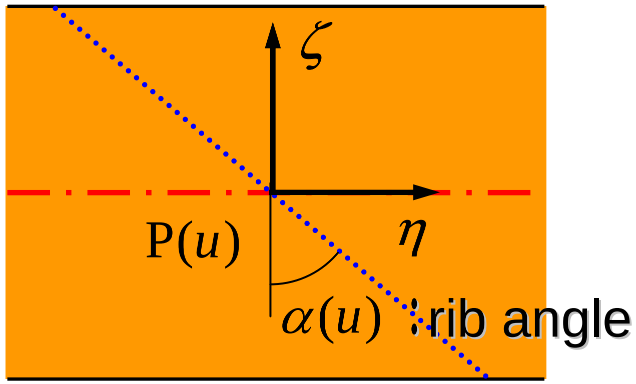
The shape of a belt object:

- Shape of the bent and twisted spine line

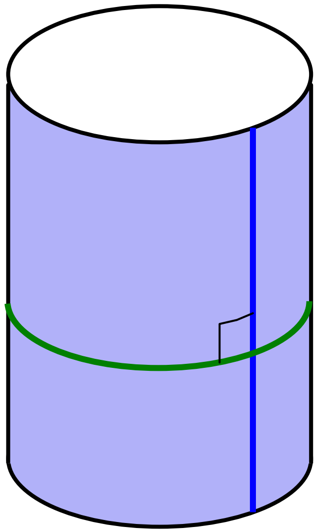
  → $\phi(u), \theta(u), \psi(u)$

- Direction of straight rib lines

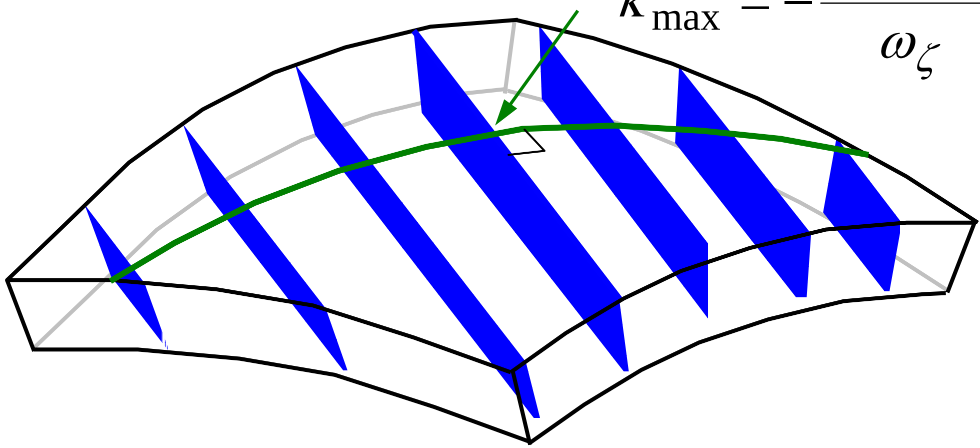
  → $\alpha(u)$



Potential Energy of Belt Object



Cylindrical surface



$$\kappa_{\max} = -\frac{\omega_\zeta^2 + \omega_\eta^2}{\omega_\zeta}$$

Potential energy : $I = \int_0^U \frac{R_f}{2} \kappa_{\max}^2 du = \int_0^U \frac{R_f}{2} \frac{\left(\omega_\zeta^2 + \omega_\eta^2\right)^2}{\omega_\zeta^2} du$

R_f : flexural rigidity along the spine line

Constraints

- Necessary constraints for developability
  - To maintain initial shape in uv-space :

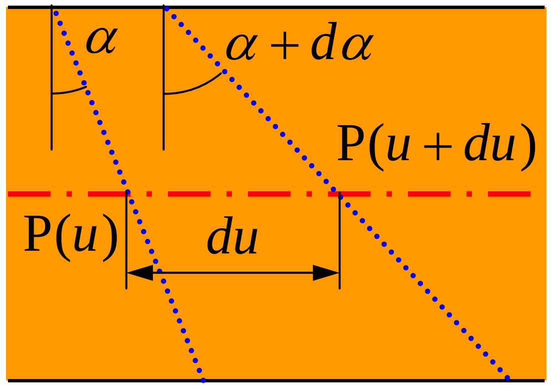
$$\omega_{u\xi} = 0,\ \forall u \in [0, U]$$

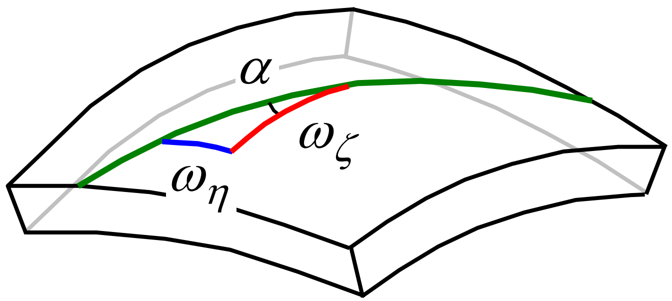
  - To prevent rib lines from intersecting with themselves :

$$-\frac{2\cos^2\alpha}{V} \le \frac{d\alpha}{du} \le \frac{2\cos^2\alpha}{V},\ \forall u \in [0, U]$$

  - Relationship between the rib angle and infinitesimal rotational angles:

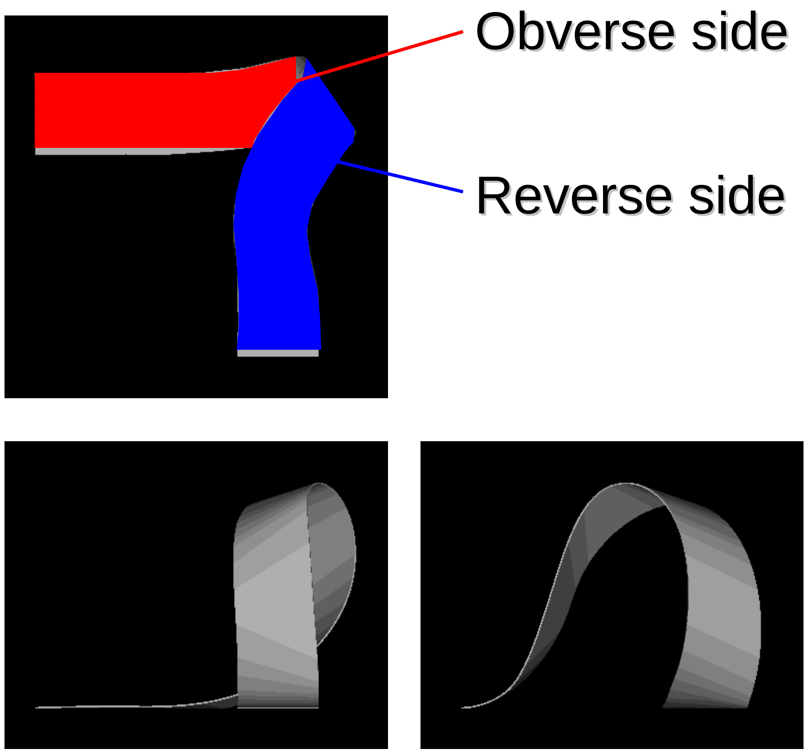
$$\alpha = -\tan^{-1}\frac{\omega_\eta}{\omega_\zeta},\ \forall u \in [0, U]$$

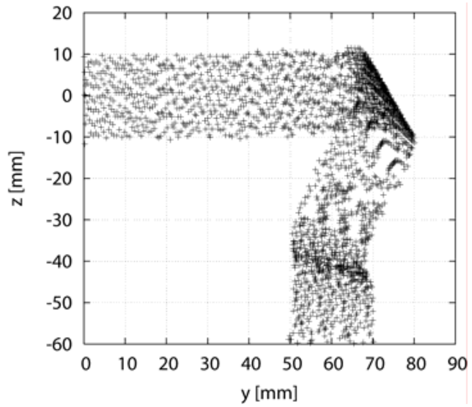




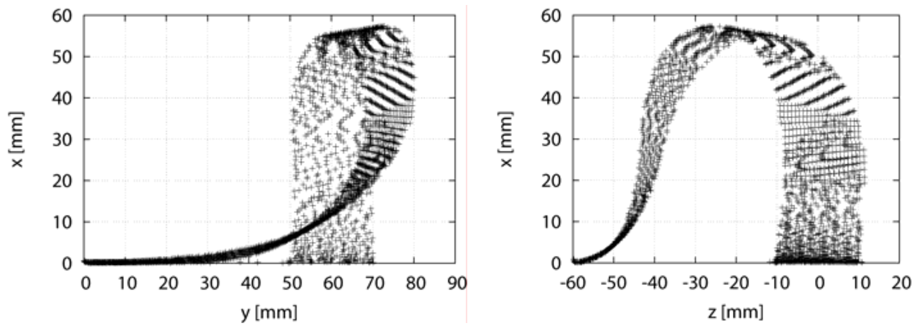
- Geometric constraints

Experimental Verification





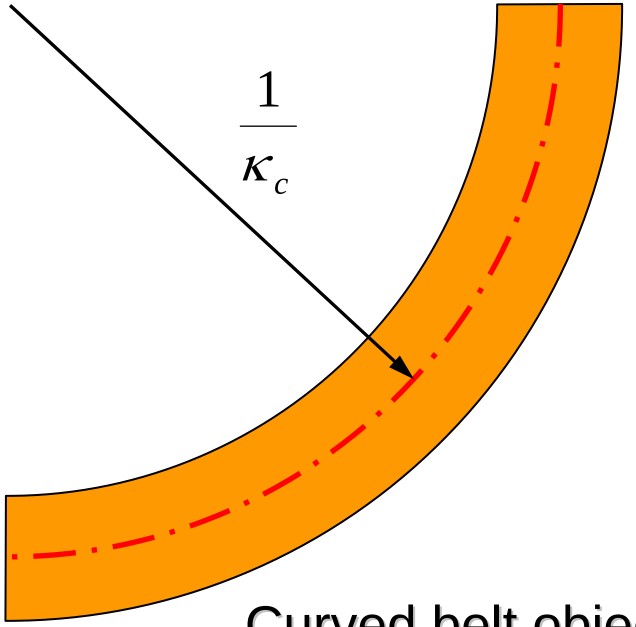
- Polystyrene
- 200[mm] long
- 20[mm] wide
- 140[μm] thick



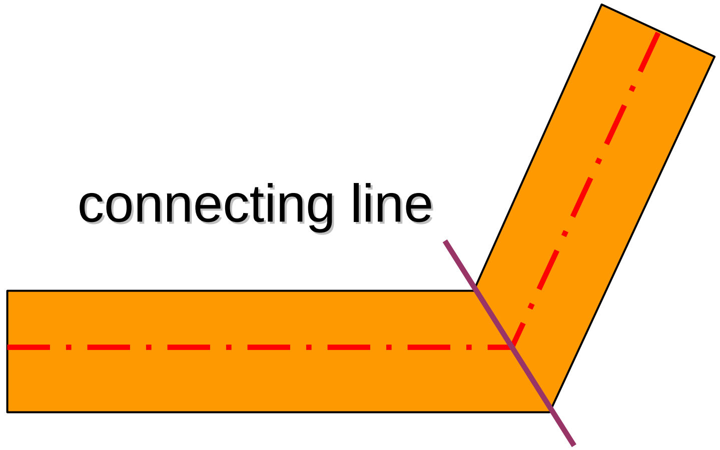
(a) Computational result

(b) Experimental result

Application to Curved/Bent Belt Object



Curved belt object



Bent belt object

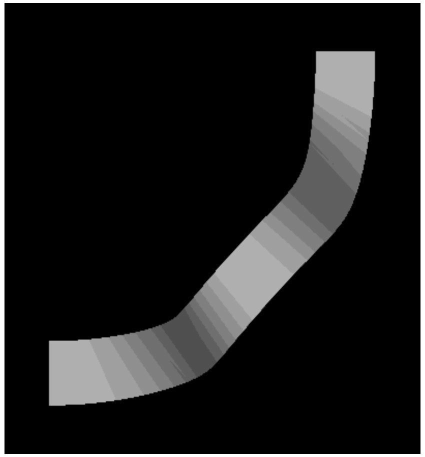
Shape in *uv*-space:

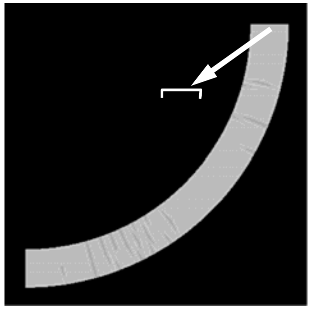
$$\omega_\xi \equiv \kappa_c$$

Assumption :

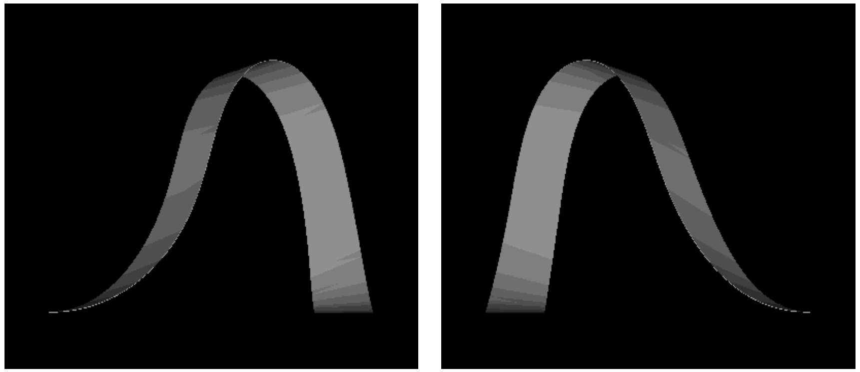
The rib line at the bent point coincides with the connecting line.

Deformed Shape of Curved Belt Object

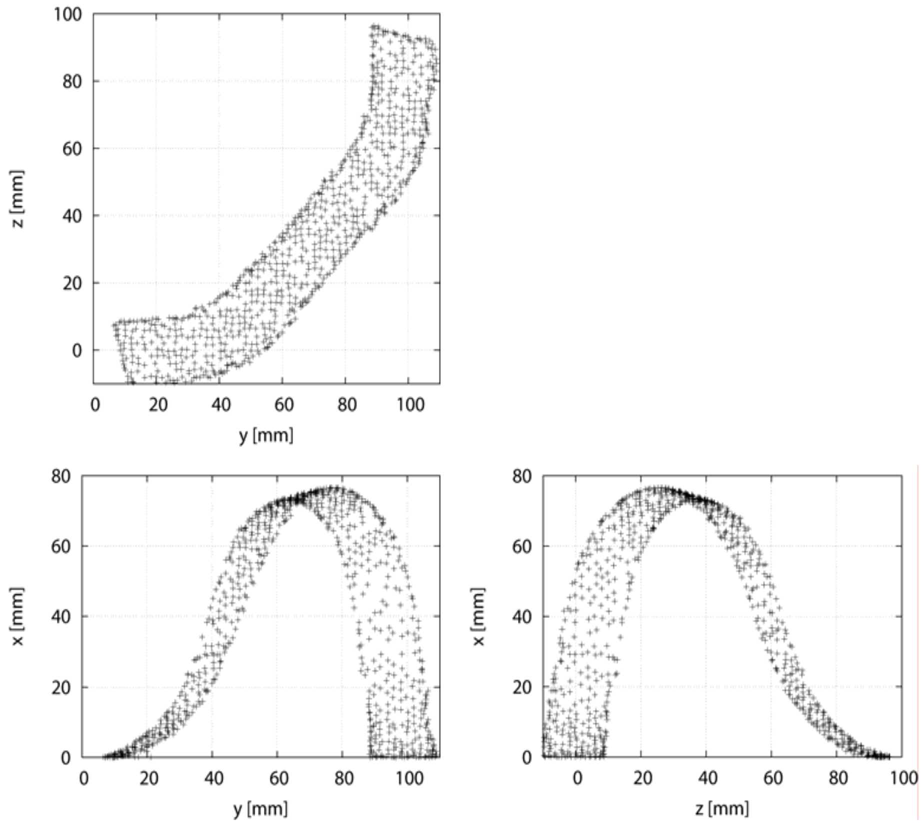




(a) Initial shape

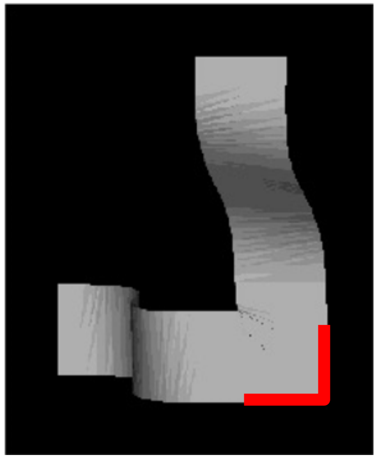


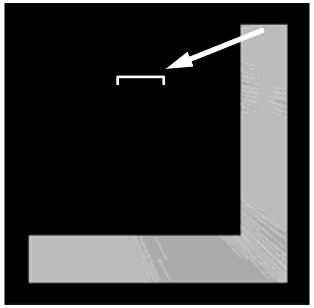
(b) Computational result



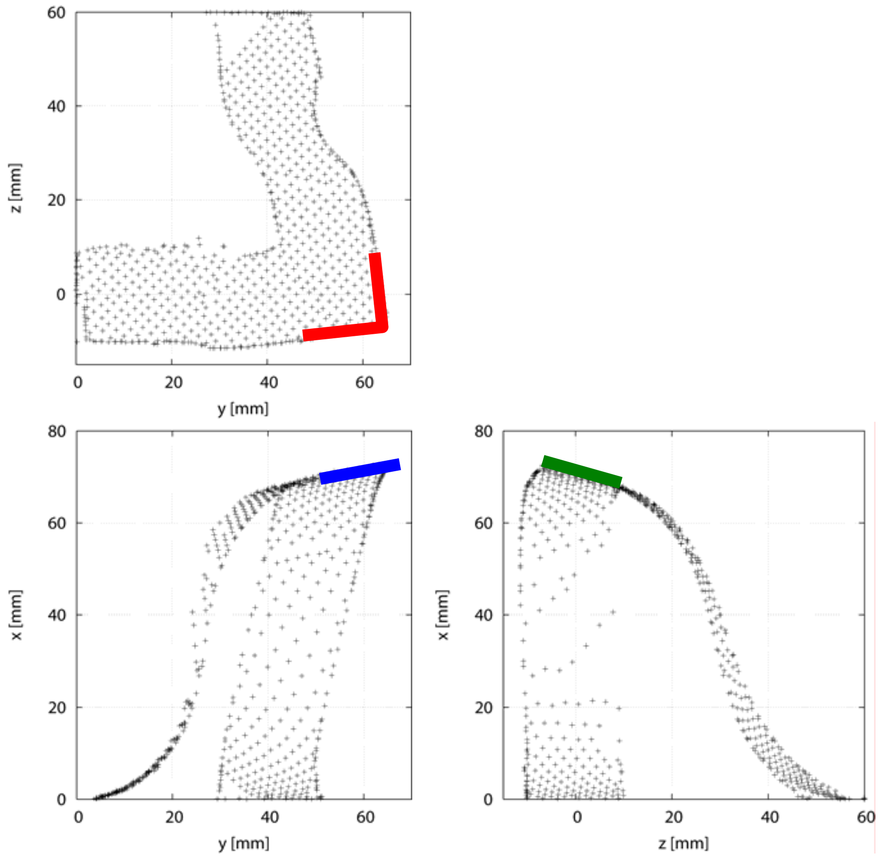
(c) Experimental result

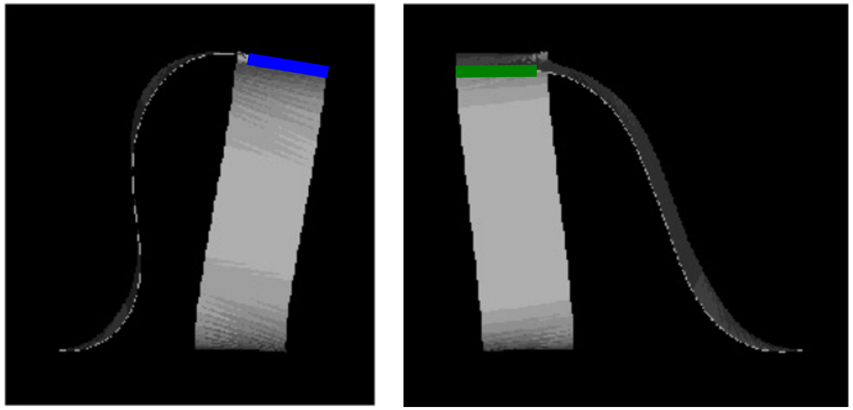
Deformed Shape of Bent Belt Object





(a) Initial shape





(b) Computational result

(c) Experimental result

Conclusions

A modeling method of linear/belt object deformation based on differential geometry was proposed.

- Differential geometry was extended to describe linear object deformation including flexure, torsion, and extension.
- The shape of a linear object can be described by four independent variables if it is extensible and by three otherwise.
- It was shown that more complex shapes such as knots and knitted fabrics also can be computed using our proposed approach.
- This approach was applied to deformation of an inextensible belt object.
- It was found that the belt object shape can be described by two independent variables.