

Analytical Mechanics

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Agenda

- 1 Schedule
- 2 Introduction to Analytical Mechanics
- 3 Illustrative Examples
 - Free fall of a mass
 - Open/Closed link mechanisms
 - Watt's governor
 - Beam deformation
- 4 MATLAB environment
- 5 Summary

Schedule (tentative)

Introduction	1 week
Variational Principles	2 weeks
Link Mechanisms	3 weeks
Rigid Body Rotation	3 weeks
Elastic Deformation	5 weeks

Web page

web page

<http://www.ritsumei.ac.jp/~hirai/>

English → Classes → 2026 Analytical Mechanics

or directly

http://www.ritsumei.ac.jp/~hirai/edu/2026/analytical_mechanics/analytical_mechanics.html

Newton mechanics vs Lagrange mechanics

Newton mechanics

vectors

linear momentum, force, angular momentum, moment, ...

vectors **depend** on coordinate systems

internal forces have to be identified and eliminated

constraints should be solved explicitly

Lagrange mechanics

scalars

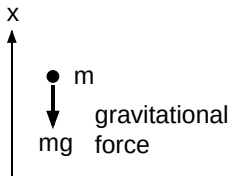
kinetic energy, potential energy, work done by external forces, ...

scalars are **independent** of coordinate systems

internal forces do not appear in Lagrangian

constraints can be incorporated into Lagrangian

Free fall of a mass (Newton mechanics)



Newton mechanics

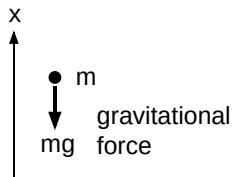
linear momentum $p = mv$

Newton's eq. of motion $\frac{dp}{dt} = -mg$

$$\frac{dp}{dt} = \frac{d}{dt}(mv) = m\dot{v}$$

differential equation $m\dot{v} = -mg$

Free fall of a mass (Lagrange mechanics)



Lagrange mechanics

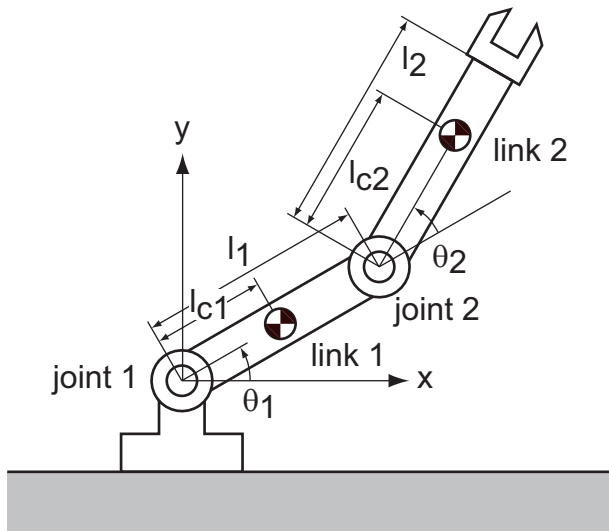
kinetic energy $T = \frac{1}{2}mv^2$

potential energy $U = mgx$

Lagrangian $\mathcal{L} = T - U = \frac{1}{2}mv^2 - mgx$

Lagrange eq. of motion $\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial v} \right) = -mg - m\dot{v} = 0$

Open link mechanism



Open link mechanism

Newton mechanics

- 1 identify all forces applied to each link (inc. internal forces)
- 2 apply Newton's eqs. of motion (and Euler's eqs. of rotation)

$$m_1 \dot{\mathbf{v}}_1 = m_1 \mathbf{g} + \mathbf{R}^{1,0} + \mathbf{R}^{1,2}, \quad m_2 \dot{\mathbf{v}}_2 = m_2 \mathbf{g} + \mathbf{R}^{2,1}, \quad \dots$$
$$I_1 \dot{\boldsymbol{\omega}}_1 = \mathbf{x}_{c1,1} \times \mathbf{R}^{1,0} + \mathbf{x}_{c1,2} \times \mathbf{R}^{1,2}, \quad I_2 \dot{\boldsymbol{\omega}}_2 = \mathbf{x}_{c2,2} \times \mathbf{R}^{2,1}$$

- 3 eliminate internal forces $\mathbf{R}^{1,0}, \mathbf{R}^{1,2}, \mathbf{R}^{2,1}$

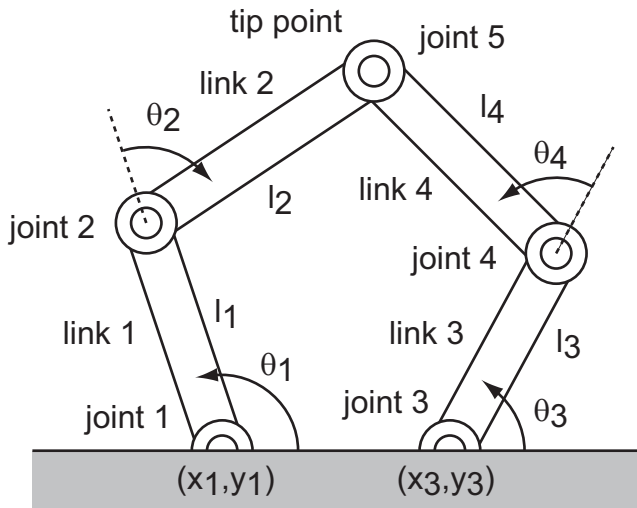
Lagrange mechanics

- 1 formulate kinetic and potential energies

$$T = T_1 + T_2, \quad U = U_1 + U_2$$

- 2 apply Lagrange's eqs. of motion to Lagrangian $\mathcal{L} = T - U$

Closed link mechanism



Closed link mechanism

left arm link 1 – link 2 $\Rightarrow$ open link mech. $\Rightarrow$ Lagrangian $\mathcal{L}_{\text{left}}$

right arm link 3 – link 4 $\Rightarrow$ open link mech. $\Rightarrow$ Lagrangian $\mathcal{L}_{\text{right}}$

geometric constraints

tip position of left arm = tip position of right arm

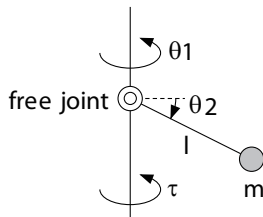
$$X \triangleq l_1 C_1 + l_2 C_{1+2} - l_3 C_3 - l_4 C_{3+4} + x_1 - x_3 = 0$$

$$Y \triangleq l_1 S_1 + l_2 S_{1+2} - l_3 S_3 - l_4 S_{3+4} + y_1 - y_3 = 0$$

Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{left}} + \mathcal{L}_{\text{right}} + \lambda_x X + \lambda_y Y$$

Watt's governor (Newton mechanics)



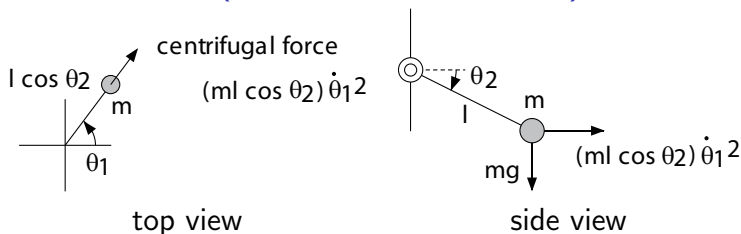
rotation around driving axis

$$I_1 = m(l \cos \theta_2)^2 = ml^2 \cos^2 \theta_2$$

$$\tau = \frac{d}{dt}(I_1 \dot{\theta}_1) = \dot{I}_1 \dot{\theta}_1 + I_1 \ddot{\theta}_1$$

$$\tau = \left\{ ml^2 \cdot 2 \cos \theta_2 (-\sin \theta_2) \dot{\theta}_2 \right\} \dot{\theta}_1 + \left\{ ml^2 \cos^2 \theta_2 \right\} \ddot{\theta}_1$$

Watt's governor (Newton mechanics)



rotation around free-joint axis

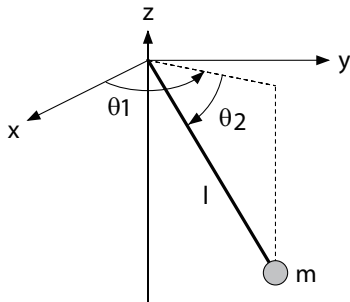
$$I_2 = ml^2$$

$$\frac{d}{dt}(I_2 \dot{\theta}_2) = mg \times l \cos \theta_2 - ml \cos \theta_2 \dot{\theta}_1^2 \times l \sin \theta_2$$

$$ml^2 \ddot{\theta}_2 = mgl \cos \theta_2 - ml^2 \cos \theta_2 \sin \theta_2 \dot{\theta}_1^2$$

need to identify centrifugal force

Watt's governor (Lagrange mechanics)



position of mass

$$\mathbf{x} = \begin{bmatrix} l \cos \theta_1 \cos \theta_2 \\ l \sin \theta_1 \cos \theta_2 \\ -l \sin \theta_2 \end{bmatrix} = l \begin{bmatrix} \cos \theta_1 \cos \theta_2 \\ \sin \theta_1 \cos \theta_2 \\ -\sin \theta_2 \end{bmatrix}$$

Watt's governor (Lagrange mechanics)

velocity of mass

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{x}}{\partial \theta_1} \frac{d\theta_1}{dt} + \frac{\partial \mathbf{x}}{\partial \theta_2} \frac{d\theta_2}{dt} \\ &= l\dot{\theta}_1 \begin{bmatrix} -\sin \theta_1 \cos \theta_2 \\ \cos \theta_1 \cos \theta_2 \\ 0 \end{bmatrix} + l\dot{\theta}_2 \begin{bmatrix} -\cos \theta_1 \sin \theta_2 \\ -\sin \theta_1 \sin \theta_2 \\ -\cos \theta_2 \end{bmatrix} \\ v^2 &= (l\dot{\theta}_1)^2 \cdot \cos^2 \theta_2 + (l\dot{\theta}_2)^2 \cdot 1 + 2(l\dot{\theta}_1)(l\dot{\theta}_2) \cdot 0 \\ &= l^2(\cos^2 \theta_2 \dot{\theta}_1^2 + \dot{\theta}_2^2)\end{aligned}$$

kinetic/potential energies, work done by external torque

$$T = \frac{1}{2}ml^2(\cos^2 \theta_2 \dot{\theta}_1^2 + \dot{\theta}_2^2), \quad U = -mgl \sin \theta_2, \quad W = \tau \theta_1$$

Watt's governor (Lagrange mechanics)

Lagrangian

$$\mathcal{L} \triangleq T - U + W$$

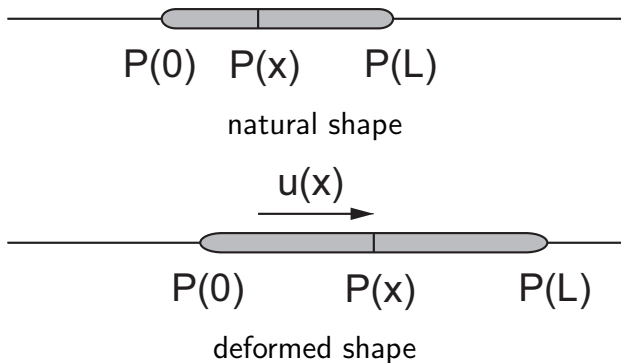
Lagrange eqs. of motion

$$\frac{\partial \mathcal{L}}{\partial \theta_k} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_k} \right) = 0, \quad (k = 1, 2)$$

$$\begin{aligned} \tau - \left\{ ml^2 \cdot 2 \cos \theta_2 (-\sin \theta_2) \dot{\theta}_2 \right\} \dot{\theta}_1 - \left\{ ml^2 \cos^2 \theta_2 \right\} \ddot{\theta}_1 &= 0 \\ -ml^2 \cos \theta_2 \sin \theta_2 \dot{\theta}_1^2 + mgl \cos \theta_2 - ml^2 \ddot{\theta}_2 &= 0 \end{aligned}$$

centrifugal or Coriolis terms yield naturally

Beam deformation



Deformation is described by function $u(x)$ ($0 \leq x \leq L$)

Beam deformation

elastic potential energy

$$U = \int_0^L \frac{1}{2} EA \left(\frac{du}{dx} \right)^2 dx$$

piecewise linear approximation

dividing interval $[0, L]$ into 6 regions:

$$\int_0^L dx = \int_{x_0}^{x_1} dx + \int_{x_1}^{x_2} dx + \cdots + \int_{x_5}^{x_6} dx$$

linear approximation:

$$\int_{x_i}^{x_j} \frac{1}{2} EA \left(\frac{du}{dx} \right)^2 dx \approx \frac{1}{2} \begin{bmatrix} u_i & u_j \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix}$$

Beam deformation

elastic potential energy

$$U = \frac{1}{2} \begin{bmatrix} u_0 & u_1 & \cdots & u_6 \end{bmatrix} \frac{EA}{h} \begin{bmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 1 & \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_5 \\ u_6 \end{bmatrix}$$

Deformation is described by a finite number of variables u_0 through u_6
finite element method (FEM)

What is MATLAB?

- ① Software for numerical calculation
- ② can handle vectors or matrices directly
- ③ Functions such as ODE solvers and optimization
- ④ Toolboxes for various applications
- ⑤ both programming and interactive calculation

What is MATLAB?

MATLAB environment

MATLAB Total Academic Headcount (TAH)

MATLAB with all toolboxes is available

Information

<https://it.support.ritsumei.ac.jp/hc/ja>

What is MATLAB?

- Install MATLAB into your own PC or mobile
- Sample programs are on the web of the class
- You can use your own PC or mobile in class

MATLAB Onramp

MATLAB Onramp (English)

<https://matlabacademy.mathworks.com/en>

MATLAB Onramp (Japanese)

<https://matlabacademy.mathworks.com/jp>

Summary: pros & cons of Lagrange mechanics

Pros

- scalar description
- once energies and works are formulated, derivative calculation yields equations of motion directly
- do not have to introduce internal forces
- effective for complex systems, such as link mechanisms, rotating or deforming objects

Cons

- difficult to understand the derived equation intuitively
- all non-potential forces, such as friction and viscous forces, are treated as external forces