Analytical Mechanics: Variational Principles

Shinichi Hirai

Dept. Robotics, Ritsumeikan Univ.

Agenda

Statics

Variation principle in statics

$$\text{minimize} \quad I = U - W$$

under constraint

$$\text{minimize} \quad I = U - W$$
$$\text{subject to} \quad R = 0$$

Solutions

- analytically solve $\delta I = 0$
- numerical optimization (`fminbnd` or `fmincon`)

Statics in variational form

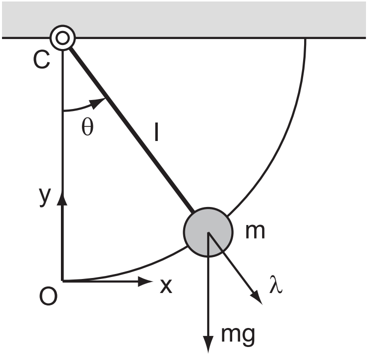
U	potential energy
W	work done by external forces/torques

Variational principle in statics

Internal energy $I = U - W$ reaches to its minimum at equilibrium:

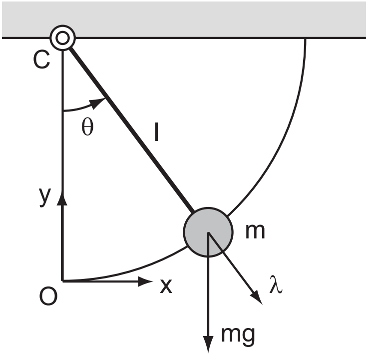
$$I = U - W \rightarrow \text{minimum}$$

Example (simple pendulum)



simple pendulum of length *l* and mass *m* suspended at point C
τ: external torque around C, θ: angle around C
Given τ, derive θ at equilibrium.

Example (simple pendulum)



$$U = mgl(1 - \cos\theta), \quad W = \tau\theta$$
$$I = mgl(1 - \cos\theta) - \tau\theta$$

Example (simple pendulum)

Solve

$$\delta I = 0$$

analytically

$$\Downarrow$$

$$I = mgl(1 - \cos\theta) - \tau\theta$$
$$I + \delta I = mgl(1 - \cos(\theta + \delta\theta)) - \tau(\theta + \delta\theta)$$

Example (simple pendulum)

Note that $\cos(\theta + \delta\theta) = \cos\theta - (\sin\theta)\delta\theta$:

$$I = mgl(1 - \cos\theta) - \tau\theta$$
$$I + \delta I = mgl(1 - \cos\theta + (\sin\theta)\delta\theta) - \tau(\theta + \delta\theta)$$

$$\Downarrow$$

$$\begin{aligned} \delta I &= mgl(\sin\theta)\delta\theta - \tau\delta\theta \\ &= (mgl\sin\theta - \tau)\delta\theta \equiv 0, \qquad \forall\delta\theta \end{aligned}$$

$$\Downarrow$$

$$mgl\sin\theta - \tau = 0$$

Example (simple pendulum)

Solve

$$\text{minimize} \quad I = mgl(1 - \cos\theta) - \tau\theta$$
$$(-\pi \leq \theta \leq \pi)$$

numerically

$$\Downarrow$$

Apply `fminbnd` to minimize a function numerically

Example (simple pendulum)

Sample Programs

- minimizing internal energy
- internal energy of simple pendulm

Example (simple pendulum)

Result

```
>> internal_energy_simple_pendulum_min

thetamin =

    0.5354


Imin =

   -0.0261
```

Statics under single constraint

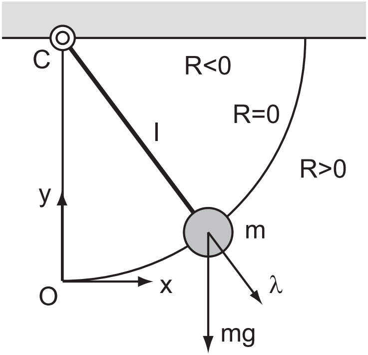
U	potential energy
W	work done by external forces/torques
R	geometric constraint

Variational principle in statics

Internal energy $U - W$ reaches to its minimum at equilibrium under geometric constraint $R = 0$:

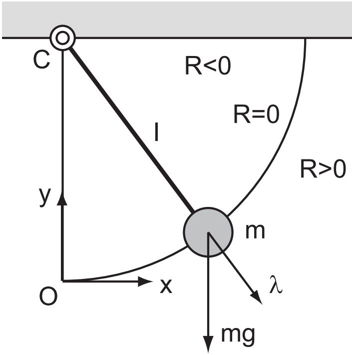
$$\begin{aligned} \text{minimize} \quad & U - W \\ \text{subject to} \quad & R = 0 \end{aligned}$$

Example (pendulum in Cartesian coordinates)



simple pendulum of length l and mass m suspended at point C
$[\, x,\ y \,]^{\top}$: position of mass
$[\, f_x,\ f_y \,]^{\top}$: external force applied to mass
Given $[\, f_x,\ f_y \,]^{\top}$, derive $[\, x,\ y \,]^{\top}$ at equilibrium.

Example (pendulum in Cartesian coordinates)



geometric constraint

distance between C and mass $= l$

$$R \triangleq \left\{x^2 + (y-l)^2\right\}^{1/2} - l = 0$$

Statics under single constraint

Solve

$$\begin{aligned} &\text{minimize} \quad U - W \\ &\text{subject to} \quad R = 0 \end{aligned}$$

analytically

$$\Downarrow$$

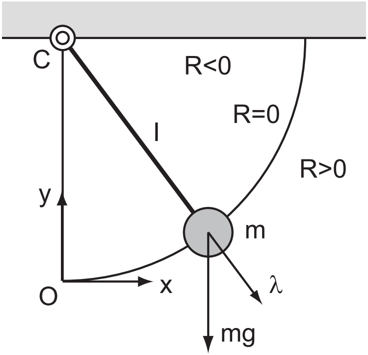
$$\text{minimize} \quad I = U - W - \lambda R$$

λ: Lagrange's multiplier

$$\Downarrow$$

$$\delta I = \delta(U - W - \lambda R) = 0$$

Example (pendulum in Cartesian coordinates)



$$U = mgy, \quad W = f_x x + f_y y$$

$$R = \left\{ x^2 + (y - l)^2 \right\}^{1/2} - l$$

Example (pendulum in Cartesian coordinates)

$$I = mgy - (f_x x + f_y y) - \lambda \left[\left\{ x^2 + (y - l)^2 \right\}^{1/2} - l \right]$$

Note that $\delta R = R_x \, \delta x + R_y \, \delta y$, where

$$R_x \triangleq \frac{\partial R}{\partial x} = x \left\{ x^2 + (y - l)^2 \right\}^{-1/2}$$
$$R_y \triangleq \frac{\partial R}{\partial y} = (y - l) \left\{ x^2 + (y - l)^2 \right\}^{-1/2}$$

$$\Downarrow$$

$$\begin{aligned} \delta I &= mg \, \delta y - f_x \delta x - f_y \delta y - \lambda R_x \delta x - \lambda R_y \delta y \\ &= (-f_x - \lambda R_x)\delta x + (mg - f_y - \lambda R_y)\delta y \equiv 0, \qquad \forall \delta x, \delta y \end{aligned}$$

Example (pendulum in Cartesian coordinates)

$$-f_x - \lambda R_x = 0$$
$$mg - f_y - \lambda R_y = 0$$

$$\Downarrow$$

$$\begin{bmatrix} 0 \\ -mg \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \end{bmatrix} + \lambda \begin{bmatrix} R_x \\ R_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ -mg \end{bmatrix}$ grav. force, $\begin{bmatrix} f_x \\ f_y \end{bmatrix}$ ext. force, $\lambda \underbrace{\begin{bmatrix} R_x \\ R_y \end{bmatrix}}_{\text{gradient vector } (\perp \text{ to } R = 0)}$ constraint force

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x, y, and λ:

$$\begin{aligned} -f_x - \lambda R_x &= 0 \\ mg - f_y - \lambda R_y &= 0 \\ R &= 0 \end{aligned}$$

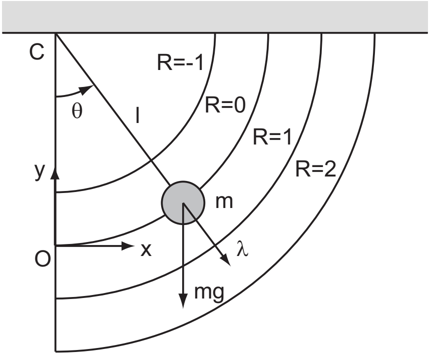
$$\Downarrow$$

we can determine position of mass $[\, x,\ y \,]^\top$ and magnitude of constraint force λ

Example (pendulum in Cartesian coordinates)

Note

$$
\begin{aligned}
I &= U - W - \lambda R \\
&= U - (W + \lambda R)
\end{aligned}
$$



λ	magnitude of a constraint force
R	distance along the force

constraint force $\perp$
contour $R =$ constant

λR	work done by a constraint force

$W + \lambda R$ work done by external & constraint forces

Statics under single constraint

Solve

$$\text{minimize} \quad I = U - W$$
$$\text{subject to} \quad R = 0$$

numerically

$$\Downarrow$$

Apply `fmincon` to minimize a function numerically under constraints
Note: "Optimization Toolbox" is needed to use `fmincon`

Example (pendulum in Cartesian coordinates)

Sample Programs

- minimizing internal energy (Cartesian)
- internal energy of simple pendulm (Cartesian)
- constraints

Example (pendulum in Cartesian coordinates)

Result:

```
>> internal_energy_pendulum_Cartesian_min
Local minimum found that satisfies the constraints.

<stopping criteria details>

qmin =
    1.4001
    3.4281

Imin =
   -0.4897
```

Statics under multiple constraints

U	potential energy
W	work done by external forces/torques
R_1, R_2	geometric constraints

Variational principle in statics

Internal energy $U - W$ reaches to its minimum at equilibrium under geometric constraints $R_1 = 0$ and $R_2 = 0$:

$$\text{minimize} \quad U - W$$
$$\text{subject to} \quad R_1 = 0, \quad R_2 = 0$$

$$\Downarrow$$

$$\delta I = \delta(U - W - \lambda_1 R_1 - \lambda_2 R_2) = 0$$

λ_1, λ_2: Lagrange's multipliers

Dynamics

Lagrangian

$$\mathcal{L} = T - U + W$$
$$\mathcal{L} = T - U + W + \lambda R \qquad \text{(under constraint)}$$

Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \mathbf{0}$$

Solutions

- numerical ODE solver (`ode45`)
- constraint stabilization method (CSM)

Dynamics in variational form

Lagrangian $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ w.r.t.
a set of generalized coordinates $\boldsymbol{q}$ and its time derivative $\dot{\boldsymbol{q}}$
time integral of Lagrangian:

$$\text{action integral} = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\,\mathrm{d}t$$

Variational principle in dynamics

variation of action integral vanishes
for any geometrically admissible variation of $\boldsymbol{q}$

$$\text{V.I.} \stackrel{\triangle}{=} \int_{t_1}^{t_2} \delta\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\,\mathrm{d}t \quad \equiv \quad 0$$

for any $\delta\boldsymbol{q}$ satisfying $\delta\boldsymbol{q}(t_1) = 0$ and $\delta\boldsymbol{q}(t_2) = 0$

Dynamics in variational form

Lagrangian $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ w.r.t.
a set of generalized coordinates $\boldsymbol{q}$ and its time derivative $\dot{\boldsymbol{q}}$
time integral of Lagrangian:

$$\text{action integral} = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\,\mathrm{d}t$$

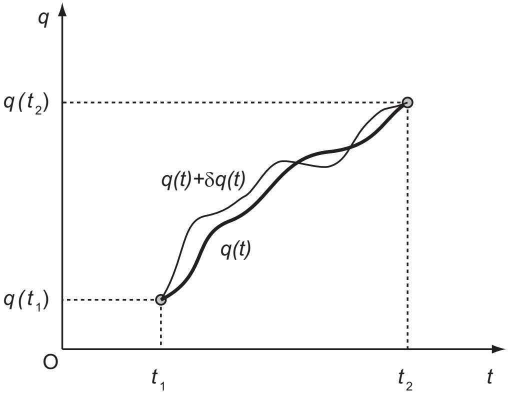
Variational principle in dynamics

variation of action integral vanishes
for any geometrically admissible variation of $\boldsymbol{q}$

$$\text{V.I.} \stackrel{\triangle}{=} \int_{t_1}^{t_2} \delta\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\,\mathrm{d}t \quad \equiv \quad 0$$

for any $\delta\boldsymbol{q}$ satisfying $\delta\boldsymbol{q}(t_1) = 0$ and $\delta\boldsymbol{q}(t_2) = 0$

Dynamics in variational form



Lagrangian corresponding to $\boldsymbol{q} + \delta \boldsymbol{q}$:

$$\mathcal{L}(\boldsymbol{q} + \delta \boldsymbol{q}, \dot{\boldsymbol{q}} + \delta \dot{\boldsymbol{q}}, t) = \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \left(\frac{\partial L}{\partial \boldsymbol{q}} \right)^{\top} \delta \boldsymbol{q} + \left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right)^{\top} \delta \dot{\boldsymbol{q}}$$

Dynamics in variational form

variation of $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$:

$$\delta \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} \right)^{\top} \delta \boldsymbol{q} + \left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \right)^{\top} \delta \dot{\boldsymbol{q}}$$

time integral of the second term:

$$\begin{aligned}
\int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \right)^{\top} \delta \dot{\boldsymbol{q}} \, \mathrm{d}t &= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \right)^{\top} \frac{\mathrm{d}}{\mathrm{d}t} \delta \boldsymbol{q} \, \mathrm{d}t \\
&= \underbrace{\left[\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \right)^{\top} \delta \boldsymbol{q} \right]_{t=t_1}^{t=t_2}}_{0 \ \ \text{since } \delta \boldsymbol{q}(t_1) = 0 \text{ and } \delta \boldsymbol{q}(t_2) = 0} - \int_{t_1}^{t_2} \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \right)^{\top} \delta \boldsymbol{q} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left(- \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \right)^{\top} \delta \boldsymbol{q} \, \mathrm{d}t
\end{aligned}$$

Dynamics in variational form

Variation of action integral:

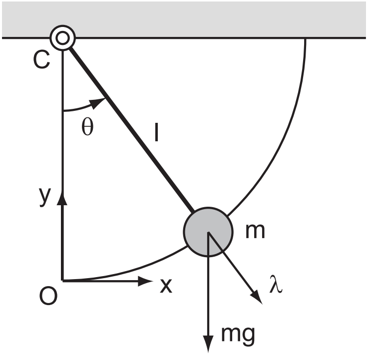
$$\begin{aligned}
\text{V.I.} &= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} \right)^{\top} \delta \boldsymbol{q} \, \mathrm{d}t + \int_{t_1}^{t_2} \left(-\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \right)^{\top} \delta \boldsymbol{q} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \right)^{\top} \delta \boldsymbol{q} \, \mathrm{d}t \quad \equiv \quad 0 \qquad \forall \delta \boldsymbol{q}
\end{aligned}$$

$$\Downarrow$$

Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \mathbf{0}$$

Example (simple pendulum)



simple pendulum of length l and mass m suspended at point C
τ: external torque around C at time t, θ: angle around C at time t
Derive the motion of the pendulum.

Dynamics in variational form

T	kinetic energy
U	potential energy
W	work done by external forces/torques

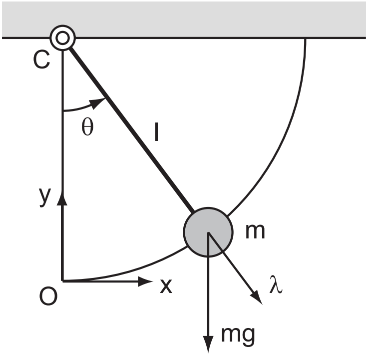
Lagrangian

$$\mathcal{L} = T - U + W$$

Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0$$

Example (simple pendulum)



$$T = \frac{1}{2}(ml^2)\dot{\theta}^2$$
$$U = mgl(1 - \cos\theta), \quad W = \tau\theta$$

Example (simple pendulum)

Lagrangian

$$\mathcal{L} = \frac{1}{2}(ml^2)\dot{\theta}^2 - mgl(1 - \cos\theta) + \tau\theta$$

partial derivatives

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta + \tau, \quad \frac{\partial L}{\partial \dot{\theta}} = (ml^2)\dot{\theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta}$$

Lagrange equation of motion

$$-mgl\sin\theta + \tau - ml^2\ddot{\theta} = 0$$

Example (simple pendulum)

Equation of the pendulum motion

$$ml^2\ddot{\theta} = -mgl\sin\theta + \tau$$

$$\Downarrow$$

Canonical form of ordinary differential equation

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{ml^2}\left(\tau - mgl\sin\theta\right)\end{aligned}$$

can be solved numerically by an ODE solver

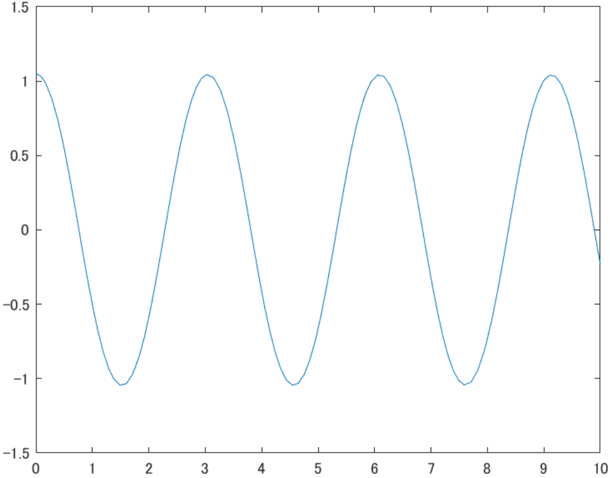
Example (simple pendulum)

Sample Programs

- solve the equation of motion of simple pendulum
- equation of motion of simple pendulum
- external torque

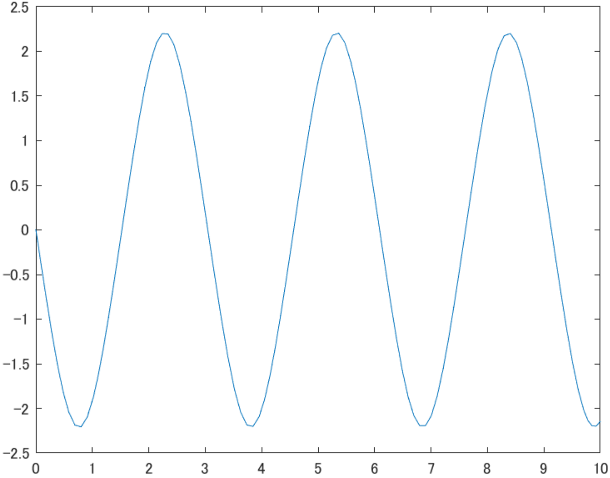
Example (simple pendulum)

Result



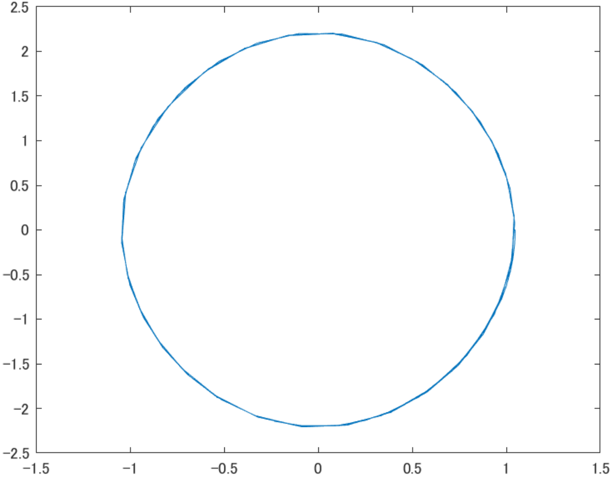
Example (simple pendulum)

Result



Example (simple pendulum)

Result



Example (pendulum with viscous friction)

Assumptions

viscous friction around supporting point C works
viscous friction causes a negative torque around C
magnitude of the torque is proportional to angular velocity

$$\text{viscous friction torque} = -b\dot{\theta} \quad (b\text{: positive constant})$$

Replacing τ by $\tau - b\dot{\theta}$:

$$(ml^2)\ddot{\theta} = (\tau - b\dot{\theta}) - mgl\sin\theta$$

$$\Downarrow$$

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \frac{1}{ml^2}\left(\tau - b\omega - mgl\sin\theta\right)$$

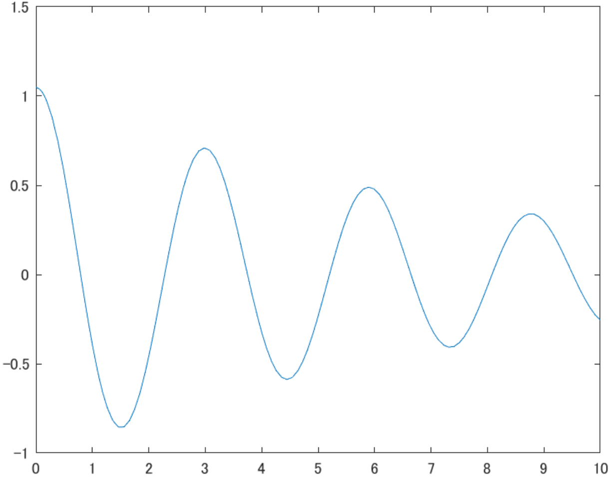
Example (pendulum with viscous friction)

Sample Programs

- solve the equation of motion of damped pendulum
- equation of motion of damped pendulum
- external torque

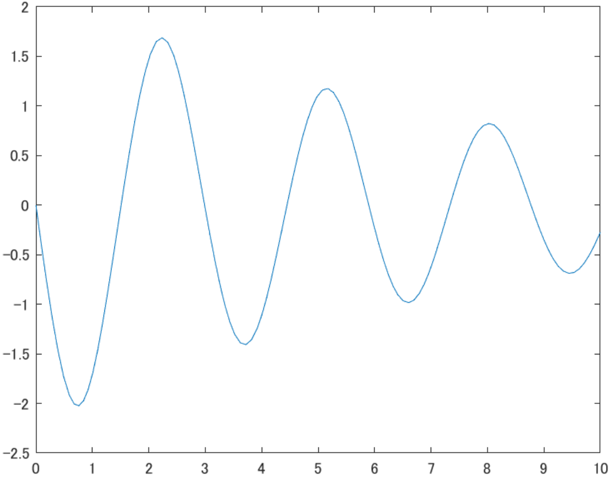
Example (pendulum with viscous friction)

Result



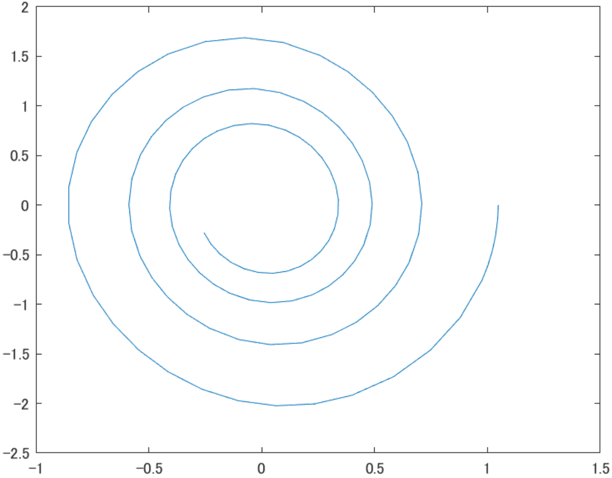
Example (pendulum with viscous friction)

Result

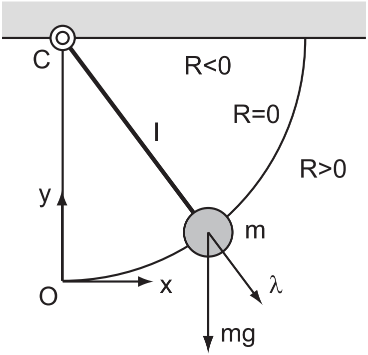


Example (pendulum with viscous friction)

Result

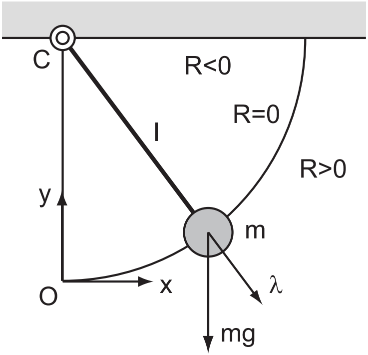


Example (pendulum in Cartesian coordinates)



simple pendulum of length l and mass m suspended at point C
$[\, x, \, y \,]^\top$: position of mass at time t
$[\, f_x, \, f_y \,]^\top$: external force applied to mass at time t
Derive the motion of the pendulum in Cartesian coordinates.

Example (pendulum in Cartesian coordinates)



geometric constraint

distance between C and mass $= l$

$$R \triangleq \left\{x^2 + (y - l)^2\right\}^{1/2} - l = 0$$

Dynamics under single constraint

T	kinetic energy
U	potential energy
W	work done by external forces/torques

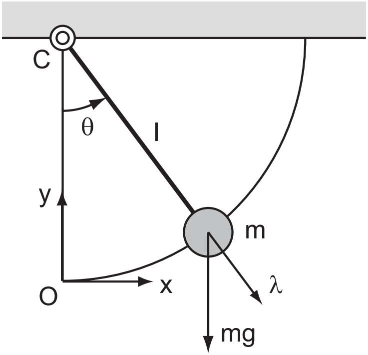
Lagrangian

$$\mathcal{L} = T - U + W + \lambda R$$

Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0$$

Example (pendulum in Cartesian coordinates)



$$T = \frac{1}{2} m \{ \dot{x}^2 + \dot{y}^2 \}$$

$$U = mgy, \quad W = f_x x + f_y y$$

$$R = \left\{ x^2 + (y - l)^2 \right\}^{1/2} - l$$

Example (pendulum in Cartesian coordinates)

Lagrangian

$$\mathcal{L} = \frac{1}{2}m\{\dot{x}^2 + \dot{y}^2\} - mgy + f_x x + f_y y + \lambda\left[\left\{x^2 + (y-l)^2\right\}^{1/2} - l\right]$$

partial derivatives

$$\frac{\partial \mathcal{L}}{\partial x} = f_x + \lambda R_x, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$
$$\frac{\partial \mathcal{L}}{\partial y} = -mg + f_y + \lambda R_y, \quad \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y}$$

Lagrange equations of motion

$$f_x + \lambda R_x - m\ddot{x} = 0$$
$$-mg + f_y + \lambda R_y - m\ddot{y} = 0$$

Example (pendulum in Cartesian coordinates)

Lagrange equations of motion

$$\underbrace{\begin{bmatrix} 0 \\ -mg \end{bmatrix}}_{\text{gravitational}} + \underbrace{\begin{bmatrix} f_x \\ f_y \end{bmatrix}}_{\text{external}} + \underbrace{\lambda \begin{bmatrix} R_x \\ R_y \end{bmatrix}}_{\text{constraint}} + \underbrace{\left\{ -m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} \right\}}_{\text{inertial}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

dynamic equilibrium among forces

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x, y, and λ:

$$
\begin{aligned}
m\ddot{x} &= f_x + \lambda R_x \\
m\ddot{y} &= -mg + f_y + \lambda R_y \\
R &= 0
\end{aligned}
$$

Example (pendulum in Cartesian coordinates)

three equations w.r.t. three unknowns x, y, and λ:

$$
\begin{aligned}
m\ddot{x} &= f_x + \lambda R_x \\
m\ddot{y} &= -mg + f_y + \lambda R_y \\
R &= 0
\end{aligned}
$$

Mixture of differential and algebraic equations

$$\Downarrow$$

Difficult to solve the mixture of differential and algebraic equations

Constraint stabilization method (CSM)

Constraint stabilization

convert algebraic eq. to its almost equivalent differential eq.

$$\text{algebraic eq.} \quad R = 0$$

$$\Downarrow$$

$$\text{differential eq.} \quad \ddot{R} + 2\alpha\dot{R} + \alpha^2 R = 0$$

(α: large positive constant)

critical damping (converges to zero most quickly)

Constraint stabilization method (CSM)

Dynamic equation of motion under geometric constraint:

$$\text{differential eq.} \quad \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \mathbf{0}$$

$$\text{algebraic eq.} \quad R = 0$$

$$\Downarrow$$

$$\text{differential eq.} \quad \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \mathbf{0}$$

$$\text{differential eq.} \quad \ddot{R} + 2\alpha\dot{R} + \alpha^2 R = 0$$

can be solved numerically by an ODE solver.

Computing equation for constraint stabilization

Assume R depends on x and y: $R(x, y) = 0$
Differentiating $R(x, y)$ w.r.t time t:

$$\dot{R} = \frac{\partial R}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial R}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = R_x\dot{x} + R_y\dot{y}$$

Differentiating $R_x(x, y)$ and $R_y(x, y)$ w.r.t time t:

$$\dot{R}_x = \frac{\partial R_x}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial R_x}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = R_{xx}\dot{x} + R_{xy}\dot{y}$$

$$\dot{R}_y = \frac{\partial R_y}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial R_y}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = R_{yx}\dot{x} + R_{yy}\dot{y}$$

Second order time derivative:

$$\begin{aligned}\ddot{R} &= (\dot{R}_x\dot{x} + R_x\ddot{x}) + (\dot{R}_y\dot{y} + R_y\ddot{y}) \\ &= (R_{xx}\dot{x} + R_{xy}\dot{y})\dot{x} + R_x\ddot{x} + (R_{yx}\dot{x} + R_{yy}\dot{y})\dot{y} + R_y\ddot{y}\end{aligned}$$

Computing equation for constraint stabilization

Second order time derivative:

$$\ddot{R} = \begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

Equation to stabilize constraint:

$$-\begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \dot{x} & \dot{y} \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + 2\alpha(R_x \dot{x} + R_y \dot{y}) + \alpha^2 R$$

$$\Downarrow \quad v_x \triangleq \dot{x},\ v_y \triangleq \dot{y}$$

$$-\begin{bmatrix} R_x & R_y \end{bmatrix} \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} v_x & v_y \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} + 2\alpha(R_x v_x + R_y v_y) + \alpha^2 R$$

Example (pendulum in Cartesian coordinates)

Equation for stabilizing constraint $R(x, y) = 0$:

$$-R_x \dot{v}_x - R_y \dot{v}_y = C(x, y, v_x, v_y)$$

where

$$C(x, y, v_x, v_y) = \begin{bmatrix} v_x & v_y \end{bmatrix} \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} + 2\alpha(R_x v_x + R_y v_y) + \alpha^2 R$$

In this example

$$P = \{x^2 + (y - l)^2\}^{-1/2}, \quad R_x = xP, \quad R_y = (y - l)P$$
$$R_{xx} = P - x^2 P^3, \quad R_{yy} = P - (y - l)^2 P^3$$
$$R_{xy} = R_{yx} = -x(y - 1)P^3$$

Example (pendulum in Cartesian coordinates)

Combining equations of motion and equation for constraint stabilization:

$$
\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\begin{bmatrix} m & & -R_x \\ & m & -R_y \\ -R_x & -R_y & \end{bmatrix}
\begin{bmatrix} \dot{v}_x \\ \dot{v}_y \\ \lambda \end{bmatrix}
&= \begin{bmatrix} f_x \\ -mg + f_y \\ C(x, y, v_x, v_y) \end{bmatrix}
\end{aligned}
$$

five equations w.r.t. five unknown variables x, y, v_x, v_y and λ

given x, y, v_x, $v_y \Longrightarrow \dot{x}$, $\dot{y}$, $\dot{v}_x$, $\dot{v}_y$

This canonical ODE can be solved numerically by an ODE solver.

Example (pendulum in Cartesian coordinates)

Let $\boldsymbol{x} = [\,x,\ y\,]^\top$. Introducing gradient vector

$$\boldsymbol{g} = \begin{bmatrix} R_x \\ R_y \end{bmatrix}$$

yields

$$\dot{R} = \boldsymbol{g}^\top \dot{\boldsymbol{x}}$$

Introducing Hessian matrix

$$H = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix}$$

yields

$$\ddot{R} = \boldsymbol{g}^\top \ddot{\boldsymbol{x}} + \dot{\boldsymbol{x}}^\top H \dot{\boldsymbol{x}}$$

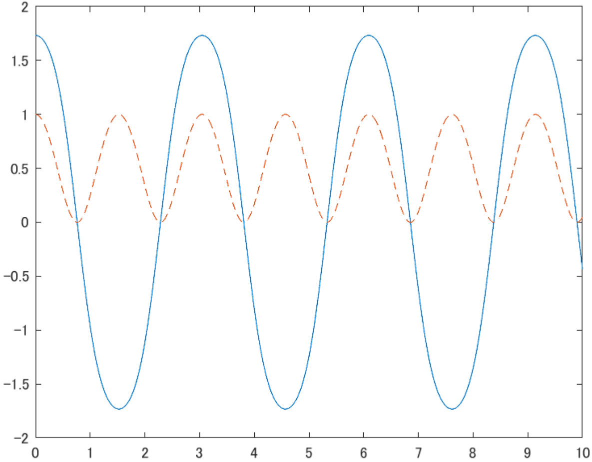
Example (pendulum in Cartesian coordinates)

Sample Programs

- solve the equation of motion of simple pendulum (Cartesian)
- equation of motion of simple pendulum (Cartesian)

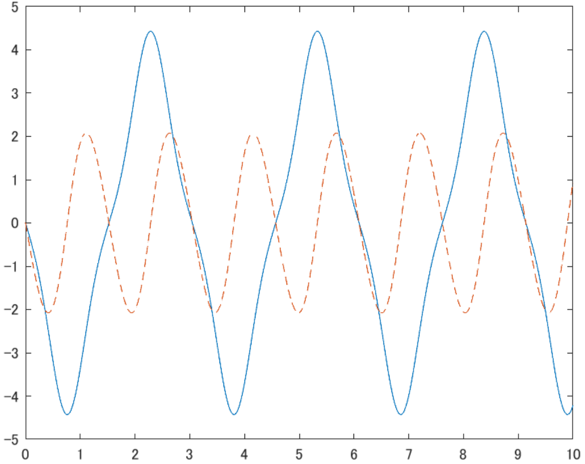
Example (pendulum in Cartesian coordinates)

t–x, y



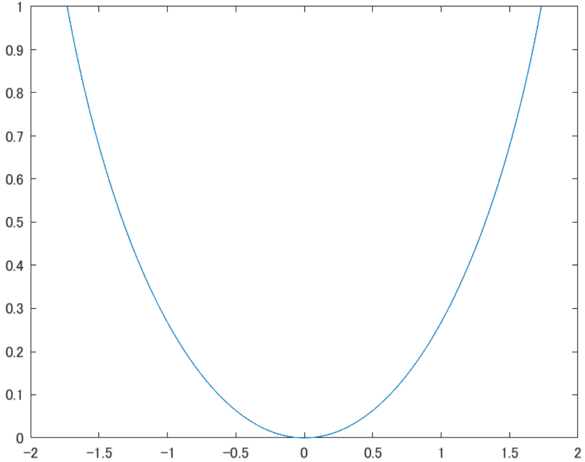
Example (pendulum in Cartesian coordinates)

t–v_x, v_y



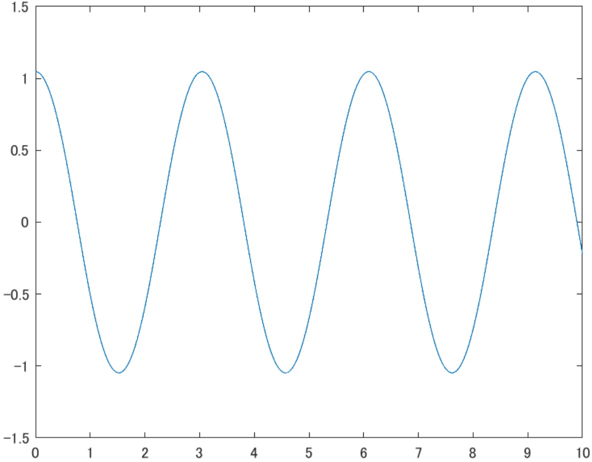
Example (pendulum in Cartesian coordinates)

x–y



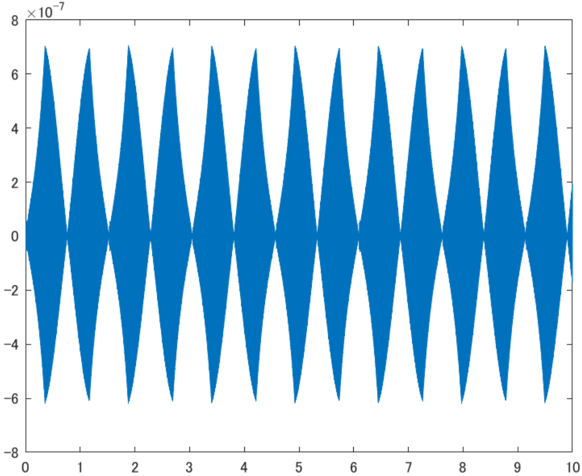
Example (pendulum in Cartesian coordinates)

t–computed θ



Example (pendulum in Cartesian coordinates)

t–constraint R



Notice

Lagrangian

$$
\begin{aligned}
\mathcal{L} &= T - U + W + \lambda R \\
&= T - (U - W - \lambda R) \\
&= T - I
\end{aligned}
$$

Lagrangian is equal to the difference between kinetic energy and internal energy under a constraint

Summary

Variational principles

- statics $I = U - W$
- statics under constraint $I = U - W - \lambda R$

$$\delta I \equiv 0$$

- dynamics $\mathcal{L} = T - U + W$
- dynamics under constraint $\mathcal{L} = T - U + W + \lambda R$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \mathbf{0}$$

- constraint stabilization method

Summary

How to solve a static problem

Solve (nonlinear) equations originated from variation

or

Numerically minimize internal energy

How to solve a dynamic problem

Step 1 Derive Lagrange equations of motion analytically

Step 2 Solve the derived equations numerically

Report

Report # 1   due date : Oct. 19 (Mon) 1:00 AM

Simulate the dynamic motion of a pendulum under viscous friction described with Cartesian coordinates x and y. Apply constraint stabilization method to convert the constraint into its almost equivalent ODE, then apply any ODE solver to solve a set of ODEs (equations of motion and equation for constraint stabilization) numerically.

Submit your report in pdf format to moodle+R

File name shoud be:
student number (11 digits) your name (without space).pdf
For example   12345678901HiraiShinichi.pdf

Report

Report # 2   due date : Oct. 26 (Mon) 1:00 AM

Assume that a system is described by four coordinates q_1 through q_4. Two constraints R_1 and R_2 are imposed on the system. Let $\boldsymbol{q} = [\, q_1,\ q_2,\ q_3,\ q_4 \,]^\top$ and $\boldsymbol{R} = [\, R_1,\ R_2 \,]^\top$. Let $\boldsymbol{g}_1$ and H_1 be gradient vector and Hessian matrix related to R_1 while $\boldsymbol{g}_2$ and H_2 be gradient vector and Hessian matrix related to R_2. Let J be Jacobian given by

$$J = \begin{bmatrix} \partial R_1/\partial q_1 & \partial R_1/\partial q_2 & \partial R_1/\partial q_3 & \partial R_1/\partial q_4 \\ \partial R_2/\partial q_1 & \partial R_2/\partial q_2 & \partial R_2/\partial q_3 & \partial R_2/\partial q_4 \end{bmatrix}$$

Show the following equations:

$$\dot{\boldsymbol{R}} = J\dot{\boldsymbol{q}}$$

$$\ddot{\boldsymbol{R}} = J\ddot{\boldsymbol{q}} + \begin{bmatrix} \dot{\boldsymbol{q}}^\top H_1 \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^\top H_2 \dot{\boldsymbol{q}} \end{bmatrix}$$