Variational Principles Appendix

Shinichi Hirai

Dept. Robotics, Ritsumeikan Univ.

Agenda

Variational calculus

Small virtual deviation of variables or functions.

$$y = x^2$$

Let us change variable x to $x + \delta x$, then variable y changes to $y + \delta y$ accordingly.

$$\begin{aligned} y + \delta y &= (x + \delta x)^2 \\ &= x^2 + 2x\,\delta x + (\delta x)^2 \\ &= x^2 + 2x\,\delta x \end{aligned}$$

Thus

$$\delta y = 2x\,\delta x$$

Variational calculus

Small virtual deviation of variables or functions.

$$I = \int_0^T \{x(t)\}^2 \, \mathrm{d}t$$

Let us change function $x(t)$ to $x(t) + \delta x(t)$, then variable I changes to $I + \delta I$ accordingly.

$$\begin{aligned} I + \delta I &= \int_0^T \{x(t) + \delta x(t)\}^2 \, \mathrm{d}t \\ &= \int_0^T \{x(t)\}^2 + 2x(t)\,\delta x(t)\,\mathrm{d}t \end{aligned}$$

Thus

$$\delta I = \int_0^T 2x(t)\,\delta x(t)\,\mathrm{d}t$$

Variational calculus

Variational operator δ

$\delta\theta$	virtual deviation of variable θ
$\delta f(\theta)$	virtual deviation of function $f(\theta)$

$$\delta f(\theta) = f'(\theta)\delta\theta$$

virtual increment of variable $\quad \theta \rightarrow \theta + \delta\theta$

increment of function
$$f(\theta) \rightarrow f(\theta + \delta\theta) = f(\theta) + f'(\theta)\delta\theta$$
$$f(\theta) \rightarrow f(\theta) + \delta f(\theta)$$

simple examples

$$\delta(5x) = 5\,\delta x \qquad \delta x^2 = 2x\,\delta x$$
$$\delta \sin\theta = (\cos\theta)\,\delta\theta, \qquad \delta\cos\theta = (-\sin\theta)\,\delta\theta$$

Variational calculus

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simple examples

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Variational calculus

assume that θ depends on time t
virtual increment of function $\theta(t) \to \theta(t) + \delta\theta(t)$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \to \frac{\mathrm{d}}{\mathrm{d}t}(\theta + \delta\theta) = \frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\delta\theta$$

$$\int \theta \, \mathrm{d}t \to \int (\theta + \delta\theta) \, \mathrm{d}t = \int \theta \, \mathrm{d}t + \int \delta\theta \, \mathrm{d}t$$

variation of derivative and integral

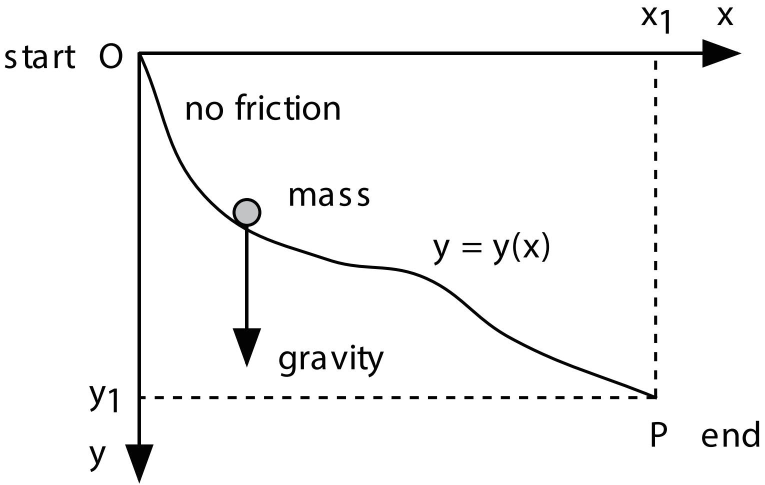
$$\delta \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\delta\theta$$

$$\delta \int \theta \, \mathrm{d}t = \int \delta\theta \, \mathrm{d}t$$

variational operator and differential/integral operator can commute

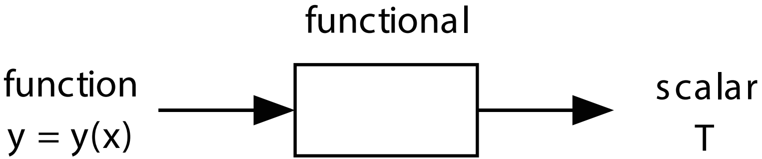
Example (curve of fastest descent)

Find a curve along that the mass falls fastest.

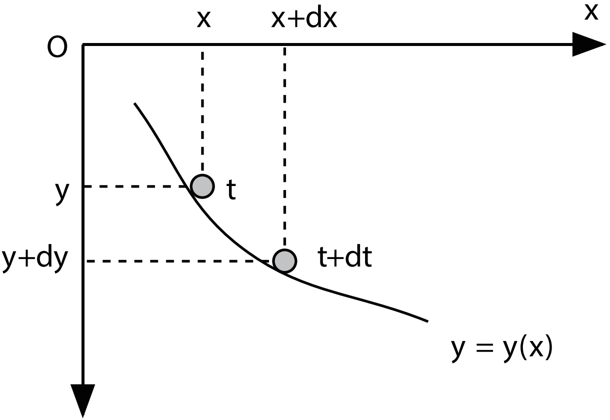


Example (curve of fastest descent)

Shape of the curve : $y(x)$
Time when the mass moves from O to P : $T = T[y]$
Scalar T depends on function y
Functional $T[y]$



Example (curve of fastest descent)



	kinetic energy	gravitational energy
point (x, y)	$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$	$-mgy$
point O	0	0

Example (curve of fastest descent)

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy = 0$$

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 2gy$$

$$(\mathrm{d}t)^2 = \frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}{2gy} = \frac{1 + (\mathrm{d}y/\mathrm{d}x)^2}{2gy}(\mathrm{d}x)^2$$

$$\mathrm{d}t = \sqrt{\frac{1 + (y')^2}{2gy}}\mathrm{d}x$$

where

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x}$$

Example (curve of fastest descent)

Consequently

$$T[y] = \int_{\text{point O}}^{\text{point P}} \mathrm{d}t = \int_0^{x_1} \sqrt{\frac{1 + (y')^2}{2gy}}\, \mathrm{d}x$$

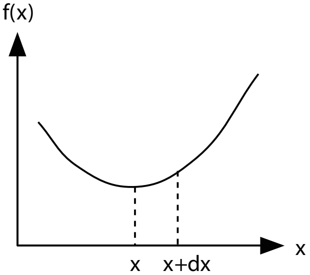
Find a function $y(x)$ that minimizes (maximizes) a functional

$$T[y] = \int_{x_0}^{x_1} F(x, y, y')\, \mathrm{d}x$$

This problem is referred to as a variational problem.

Example (curve of fastest descent)

Recall: minimization of function $f(x)$



For any deviation $\mathrm{d}x$ at the minimum point:

$$f(x + \mathrm{d}x) - f(x) = f'(x)\mathrm{d}x = 0.$$

This condition is satisfied if, and only if,

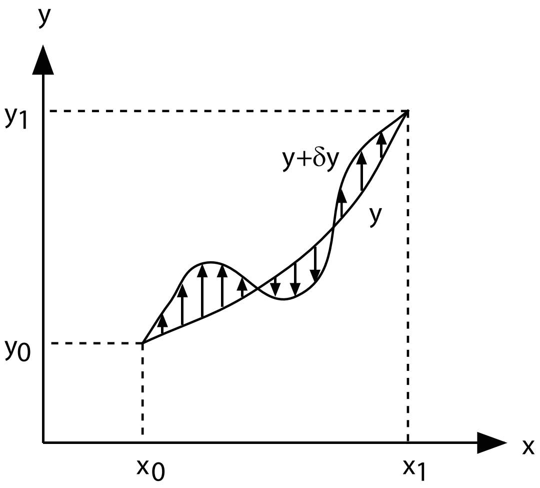
$$f'(x) = 0.$$

Solving $f'(x) = 0$ yields minimum points (local minimum points).

Example (curve of fastest descent)

Introduce the deviation of function $y(x)$ into a variational problem:

$$\delta y(x) \quad \text{where} \quad \delta y(x_0) = 0, \quad \delta y(x_1) = 0$$



Example (curve of fastest descent)

Evaluating functional $T[y]$, we have

$$T[y] = \int_{x_0}^{x_1} F(x, y, y')\,\mathrm{d}x$$

$$T[y + \delta y] = \int_{x_0}^{x_1} F(x, y + \delta y, y' + \delta y')\,\mathrm{d}x$$

Note that

$$F(x, y + \delta y, y' + \delta y') = F(x, y, y') + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

$$T[y + \delta y] - T[y] = \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right)\,\mathrm{d}x$$

Example (curve of fastest descent)

$$\int_{x_0}^{x_1} \frac{\partial F}{\partial y'} \delta y' \mathrm{d}x = \underbrace{\left[\frac{\partial F}{\partial y'} \delta y \right]_{x=x_0}^{x=x_1}}_{\substack{\uparrow \\ 0}} - \int_{x_0}^{x_1} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) \delta y \, \mathrm{d}x$$

$$T[y + \delta y] - T[y] = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) \right] \delta y \, \mathrm{d}x$$

$$y : \ \text{optimal} \quad \Leftrightarrow \quad T[y + \delta y] - T[y] = 0 \ \text{ for any } \ \delta y$$

Consequently

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) = 0$$

Euler-Lagrange's equation

Example (curve of fastest descent)

Euler-Lagrange's equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0$$

is simply described as

$$F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} = 0$$

where

$$F_y = \frac{\partial F}{\partial y}, \quad F_{y'} = \frac{\partial F}{\partial y'}$$

Example (curve of fastest descent)

In the case that $F = F(y, y')$, say, no explicit x is involved in a functional F,

$$\frac{\mathrm{d}}{\mathrm{d}x}(y'F_{y'} - F) = y''F_{y'} + y'\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}F_{y'}}_{F_y} - \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}F}_{\substack{\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x} \\ = F_y y' + F_{y'} y''}}$$

$$= 0$$

Thus, Euler-Lagrange's equation can be simplified to

$$y'F_{y'} - F = c \ \ \text{(const)}.$$

Example (curve of fastest descent)

Solving curve of fastest descent problem

$$F(y, y') = \sqrt{\frac{1 + (y')^2}{2gy}}$$

$$T[y] = \int_0^{x_1} F(y, y')\,\mathrm{d}x$$

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{(1 + y'^2)2gy}}$$

Euler-Lagrange's equation

$$y' F_{y'} - F = \frac{y'^2}{\sqrt{(1 + y'^2)2gy}} - \sqrt{\frac{1 + (y')^2}{2gy}} = c_1 \quad \text{(const)}$$

Example (curve of fastest descent)

Then,

$$
\begin{aligned}
\frac{y'^2 - (1 + y'^2)}{\sqrt{(1 + y'^2)2gy}} &= c_1 \\
-1 &= c_1 \sqrt{(1 + y'^2)2gy} \\
1 &= c_1^2 (1 + y'^2)2gy \\
y'^2 &= \frac{c_2 - y}{y} \quad \text{where} \quad c_2 = \frac{1}{2gc_1^2} \quad \text{(const)}
\end{aligned}
$$

Let

$$
y = \frac{c_2}{2}(1 - \cos u)
$$

Then,

$$
y' = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{c_2}{2} u' \sin u
$$

Example (curve of fastest descent)

$$\left(\frac{c_2}{2}u'\sin u\right)^2 = \frac{c_2 - c_2/2(1-\cos u)}{c_2/2(1-\cos u)} = \frac{1+\cos u}{1-\cos u} = \frac{\sin^2 u}{(1-\cos u)^2}$$

The above differential equation is converted into:

$$\frac{c_2}{2}u' = \frac{1}{1-\cos u}$$

$$\frac{c_2}{2}(1-\cos u)\,\mathrm{d}u = \mathrm{d}x$$

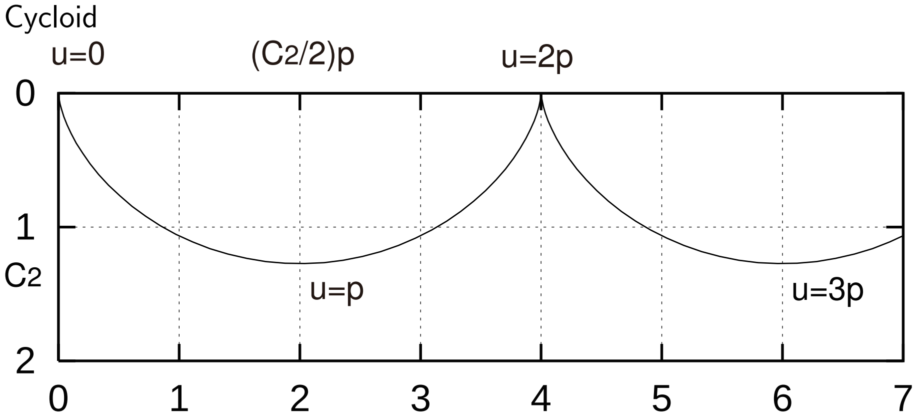
$$x = \int \frac{c_2}{2}(1-\cos u)\,\mathrm{d}u = \frac{c_2}{2}(u-\sin u) + c_3$$

Noting that $x = c_3$ and $y = 0$ at $u = 0$, we find that $c_3 = 0$. Finally, we have

$$x = \frac{c_2}{2}(u-\sin u)$$

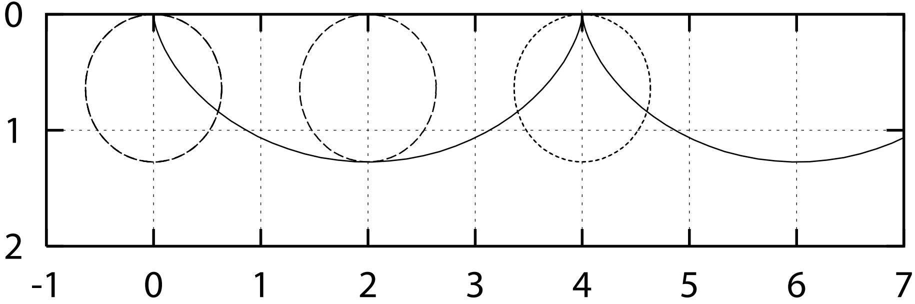
$$y = \frac{c_2}{2}(1-\cos u) \qquad \text{cycloid}$$

Example (curve of fastest descent)



Example (curve of fastest descent)

Path of point on rolling circle



Lagrange multiplier method

converts minimization (maximization) under conditions into minimization (maximization) without conditions.

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) \\ &\text{subject to} \quad g(\boldsymbol{x}) = 0 \end{aligned}$$

$$\Downarrow$$

$$\text{minimize} \quad I(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) + \lambda g(\boldsymbol{x})$$

$$\Downarrow$$

$$\begin{aligned} \frac{\partial I}{\partial \boldsymbol{x}} &= \frac{\partial f}{\partial \boldsymbol{x}} + \lambda \frac{\partial g}{\partial \boldsymbol{x}} = \boldsymbol{0} \\ \frac{\partial I}{\partial \lambda} &= g(\boldsymbol{x}) = 0 \end{aligned}$$

Lagrange multiplier method (example)

Length of each edge of a cube is given by x, y, and z.
Determine x, y, and z that minimizes the surface of the cube when the cube volume is constantly specified by a^3:

$$\text{minimize} \quad S(x, y, z) = 2xy + 2yz + 2zx$$
$$\text{subject to} \quad R(x, y, z) \stackrel{\triangle}{=} xyz - a^3 = 0$$

Introducing Lagrange multiplier λ, the above conditional minimization can be converted into the following unconditional minimization:

$$\begin{aligned} \text{minimize} \quad I(x, y, z, \lambda) &= S(x, y, z) + \lambda R(x, y, z) \\ &= 2xy + 2yz + 2zx + \lambda(xyz - a^3) \end{aligned}$$

Lagrange multiplier method (example)

Calculating partial derivatives:

$$\frac{\partial I}{\partial x} = 2y + 2z - \lambda yz = 0 \tag{1}$$

$$\frac{\partial I}{\partial y} = 2z + 2x - \lambda zx = 0 \tag{2}$$

$$\frac{\partial I}{\partial z} = 2x + 2y - \lambda xy = 0 \tag{3}$$

$$\frac{\partial I}{\partial \lambda} = xyz - a^3 = 0 \tag{4}$$

Calculating $(1) \cdot x - (2) \cdot y$, we have

$$z(x - y) = 0,$$

which directly yields $x = y$. Similarly, we have $y = z$ and $z = x$. Consequently, we concludes $x = y = z = a$.

ODE solver

Let us solve van del Pol equation:

$$\ddot{x} - 2(1 - x^2)\dot{x} + x = 0$$

Canonical form:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= 2(1 - x^2)\dot{x} - x \end{aligned}$$

State variable vector:

$$\boldsymbol{q} = \left[\begin{array}{c} x \\ v \end{array} \right]$$

ODE solver (MATLAB)

File van_der_Pol.m describes the canonical form:

```
function dotq = van_der_Pol (t,q)
    x = q(1);
    v = q(2);
    dotx = v;
    dotv = 2*(1-x^2)*v - x;
    dotq = [dotx; dotv];
end
```

File name `van_der_Pol` should conincide with function name `van_der_Pol`.

ODE solver (MATLAB)

File van_der_Pol_solve.m solves van der Pol equation numerically:

```
timestep=0.00:0.10:10.00;
qinit=[2.00;0.00];
[time,q]=ode45(@van_der_Pol,timestep,qinit);

% line style   solid -   broken -.   chain --   dotted :
plot(time,q(:,1),'-', time,q(:,2),'-.');
```

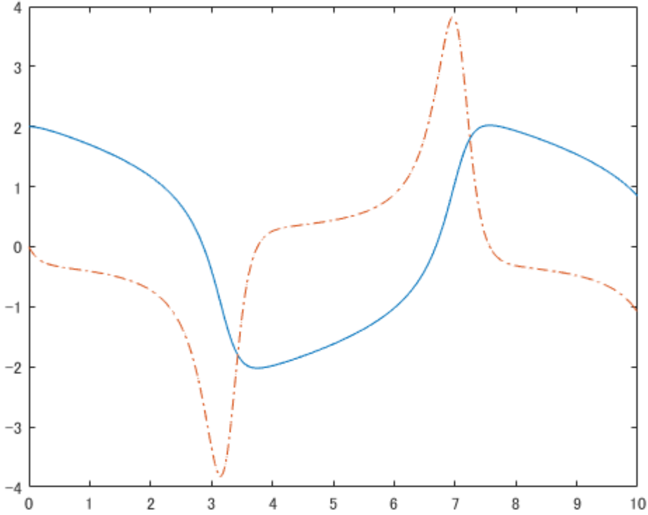
ODE solver (MATLAB)

```
>> time
time =
         0
    0.1000
    0.2000
    0.3000
    0.4000

>> q
q =
    2.0000         0
    1.9917   -0.1504
    1.9721   -0.2338
    1.9461   -0.2822
    1.9163   -0.3125
```

The first and second columns corresponds to x and v.

ODE solver (MATLAB)



Dynamics in variational form

Lagrangian $\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ w.r.t.
a set of generalized coordinates $\boldsymbol{q}$ and its time derivative $\dot{\boldsymbol{q}}$
time integral of Lagrangian:

$$\text{action integral} = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\,\mathrm{d}t$$

Variational principle in dynamics

variation of action integral vanishes
for any geometrically admissible variation of $\boldsymbol{q}$

$$\text{V.I.} \triangleq \int_{t_1}^{t_2} \delta\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\,\mathrm{d}t \quad \equiv \quad 0$$

for any $\delta\boldsymbol{q}$ satisfying $\delta\boldsymbol{q}(t_1) = 0$ and $\delta\boldsymbol{q}(t_2) = 0$

Dynamics in variational form

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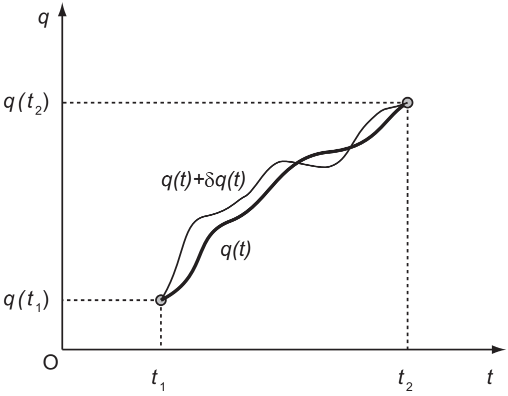
for any $\delta\boldsymbol{q}$ satisfying $\delta\boldsymbol{q}(t_1) = 0$ and $\delta\boldsymbol{q}(t_2) = 0$

Dynamics in variational form



Lagrangian corresponding to $\boldsymbol{q} + \delta\boldsymbol{q}$:

$$\mathcal{L}(\boldsymbol{q} + \delta\boldsymbol{q}, \dot{\boldsymbol{q}} + \delta\dot{\boldsymbol{q}}, t) = \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \left(\frac{\partial L}{\partial \boldsymbol{q}}\right)^\top \delta\boldsymbol{q} + \left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)^\top \delta\dot{\boldsymbol{q}}$$

Dynamics in variational form

variation of $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$:

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\right)^\top \delta\boldsymbol{q} + \left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)^\top \delta\dot{\boldsymbol{q}}$$

time integral of the second term:

$$\begin{aligned}
\int_{t_1}^{t_2} \left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)^\top \delta\dot{\boldsymbol{q}}\,\mathrm{d}t &= \int_{t_1}^{t_2} \left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)^\top \frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{q}\,\mathrm{d}t \\
&= \underbrace{\left[\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)^\top \delta\boldsymbol{q}\right]_{t=t_1}^{t=t_2}}_{0 \ \text{ since } \delta\boldsymbol{q}(t_1)=0 \text{ and } \delta\boldsymbol{q}(t_2)=0} - \int_{t_1}^{t_2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)^\top \delta\boldsymbol{q}\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)^\top \delta\boldsymbol{q}\,\mathrm{d}t
\end{aligned}$$

Dynamics in variational form

Variation of action integral:

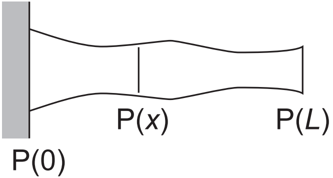
$$
\begin{aligned}
\text{V.I.} &= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} \right)^{\top} \delta \boldsymbol{q} \, \mathrm{d}t + \int_{t_1}^{t_2} \left(-\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \right)^{\top} \delta \boldsymbol{q} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \right)^{\top} \delta \boldsymbol{q} \, \mathrm{d}t \quad \equiv \quad 0 \qquad \forall \delta \boldsymbol{q}
\end{aligned}
$$

$$\Downarrow$$

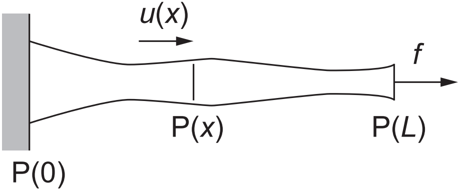
Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} = \mathbf{0}$$

Example (dynamic extension of beam)



natural shape at time 0



deformed shape at time t

Deformation at time t is described by function $u(x, t)$ $(0 \leq x \leq L)$

Example (dynamic extension of beam)

E: Young's modulus at point $\mathrm{P}(x)$
A: Cross-sectional area at point $\mathrm{P}(x)$
ρ: density at point $\mathrm{P}(x)$

Assumption: $A(x)$, $E(x)$, and $\rho(x)$ do not change despite of extension, *i.e.* axial deformation is negligible.

Kinetic energy, elastic potential energy, and work done by external force

$$T = \int_0^L \frac{1}{2}\rho A \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x$$
$$U = \int_0^L \frac{1}{2}EA \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x$$
$$W = f\, u(L,t)$$

Example (dynamic extension of beam)

Lagrangian

$$
\begin{aligned}
\mathcal{L} &= T - U + W \\
&= \int_0^L \frac{1}{2}\rho A \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x - \int_0^L \frac{1}{2} EA \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x + f\, u(L,t)
\end{aligned}
$$

Variation of Lagrangian

$$
\delta\mathcal{L} = \int_0^L \rho A \frac{\partial u}{\partial t}\frac{\partial}{\partial t}\delta u \,\mathrm{d}x - \int_0^L EA \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\delta u \,\mathrm{d}x + f\,\delta u(L,t)
$$

Recall

$$
\begin{aligned}
\delta U &= \int_0^L EA \frac{\partial u}{\partial x}\frac{\partial}{\partial x}\delta u \,\mathrm{d}x \\
&= EA \frac{\partial u}{\partial x}\delta u \bigg|_{x=L} - \int_0^L \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right)\delta u \,\mathrm{d}x
\end{aligned}
$$

Example (dynamic extension of beam)

Time-integral of the first term of $\delta\mathcal{L}$:

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_0^L \rho A \frac{\partial u}{\partial t}\frac{\partial}{\partial t}\delta u \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^L \int_{t_1}^{t_2} \rho A \frac{\partial u}{\partial t}\frac{\partial}{\partial t}\delta u \,\mathrm{d}t\,\mathrm{d}x \\
&= \int_0^L \left\{ \left[\rho A \frac{\partial u}{\partial t}\delta u \right]_{t=t_1}^{t=_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\rho A \frac{\partial u}{\partial t} \right) \delta u \,\mathrm{d}t \right\} \mathrm{d}x \\
&= \int_0^L \left\{ - \int_{t_1}^{t_2} \rho A \frac{\partial^2 u}{\partial t^2}\delta u \,\mathrm{d}t \right\} \mathrm{d}x \\
&= \int_{t_1}^{t_2}\int_0^L \left\{ -\rho A \frac{\partial^2 u}{\partial t^2}\delta u \right\} \mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

Example (dynamic extension of beam)

$$
\begin{aligned}
\mathrm{V.I.} &= \int_{t_1}^{t_2} \delta\mathcal{L}\,\mathrm{d}t \\
&= \int_{t_1}^{t_2}\int_0^L \left\{ -\rho A \frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x} \right) \right\} \delta u \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \left\{ -\, EA \frac{\partial u}{\partial x}\bigg|_{x=L} + f \right\} \delta u(L,t)\,\mathrm{d}t
\end{aligned}
$$

should be equal to 0 for any $\delta u(x,t)$

$$\Downarrow$$

$$
-\rho A \frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x} \right) = 0, \qquad -\, EA \frac{\partial u}{\partial x}\bigg|_{x=L} + f = 0
$$

Example (dynamic extension of beam)

Equation of deformation (partial differential equation)

$$\rho A \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x} \right)$$

Boundary conditions

$$u(0, t) = 0$$
$$E(L, t) A(L, t) \frac{\mathrm{d}u}{\mathrm{d}x}(L, t) = f(t)$$

Initial conditions (example)

$$u(x, 0) = 0, \quad \forall x \in [\,0,\ L\,]$$
$$\frac{\mathrm{d}u}{\mathrm{d}t}(x, 0) = 0, \quad \forall x \in [\,0,\ L\,]$$

Example (dynamic extension of beam)

Assume that E. A, and ρ are constant:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $c = \sqrt{E/\rho}$

Given function $f(x)$, let

$$u(x, t) = f(x - ct)$$

Then

$$\begin{aligned} \partial u/\partial x = f'(x - ct), \quad \partial^2 u/\partial x^2 = f''(x - ct), \\ \partial u/\partial t = f'(x - ct)(-c), \quad \partial^2 u/\partial t^2 = f''(x - ct)(-c)^2 \end{aligned}$$

Thus, $f(x - ct)$ is one solution of the PDE.

Example (dynamic extension of beam)

Given function $g(x)$, let

$$u(x,t) = g(x+ct)$$

Then

$$\begin{aligned} &\partial u/\partial x = g'(x+ct), \quad \partial^2 u/\partial x^2 = g''(x+ct), \\ &\partial u/\partial t = g'(x+ct)(+c), \quad \partial^2 u/\partial t^2 = g''(x+ct)(+c)^2 \end{aligned}$$

Thus, $g(x+ct)$ is one solution of the PDE.

Rayleigh wave

Solution $f(x-ct)$: wave propagating at speed $+c$
Solution $g(x+ct)$: wave propagating at speed $-c$